Application of possibility theory to optimization and decision making

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- Introduction to Possibility Theory
  - Possibility and necessity measures
  - Correspondence to Plausibility and Belief functions
- Application to Decision Making: Decision principles
  - Possibility measure maximization
  - Necessity measure maximization
  - Relative possibility measure maximization
- Possibilistic (Fuzzy) Linear Programming
  - (Much simpler than stochastic linear programming)

Introduction to Fuzzy Sets (1)

- **Crisp Sets**: The conventional set

(Crisp) Set A:

$x \in A$: "x belongs to A" or "x is a member of A"

$x \notin A$: "x does not belong to A" or "x is not a member of A"

Sets with unsharp boundary $\Rightarrow$ **Fuzzy Sets** (Fuzzy Subsets)

For extending crisp sets to **Fuzzy Sets**,

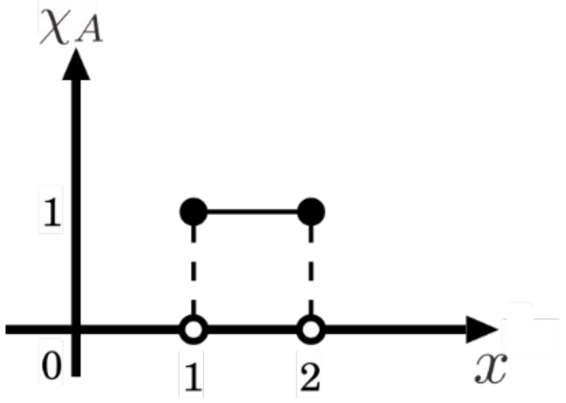
- **Characteristic Function**

The characteristic function of set A written as χ_A:

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

(Example 1)

$A = \{x | 1 \leq x \leq 2, x \in \mathbf{R}\}$



Introduction to Fuzzy Sets (2)

- **Fuzzy Sets**: Fuzzy Sets $\tilde{A}$: Proposed by Lotfi A. Zadeh in 1965
  - A set characterized by a membership function $\mu_{\tilde{A}} : \Omega \Rightarrow [0,1]$ (Ω: Universal set, A set of all objects)
  - For each $x \in \Omega$, $\mu_{\tilde{A}}(x) \in [0,1]$ is assigned
  - The closer to $\left|\begin{matrix} 1 \\ 0 \end{matrix}\right|$ $\mu_{\tilde{A}}(x) \in [0,1]$ is, the $\left|\begin{matrix} \text{higher} \\ \text{lower} \end{matrix}\right|$ the degree of membership of x to $\tilde{A}$.

Professor Lotfi Aliasker Zadeh (1921—2017)

- Born on February 4th in 1921, in Baku, Republic of Azerbaijan.
- Move to Tehran, Iran in 1931
- Bachelor in Electric Engineering from Teheran Univ. in 1946
- Master in Electric Engineering from MIT in 1946
- PhD in Electric Engineering from Columbia Univ. in 1950
- Associate Professor, Columbia Univ. in 1950 and then Professor in 1959
- Professor, UCB in 1963
- “Linear System Theory: The State Space Approach” with Prof. Desoer
- “Fuzzy Sets” in Information and Control in 1965
- Honda Prize, IEEE Richard W. Hamming Medal, IEEE Medal of Honor, The Golden Goose Award
- Passed away on September 6, 2017 in Berkley, CA, USA

Photograph from RSCTC 2006 in Kobe

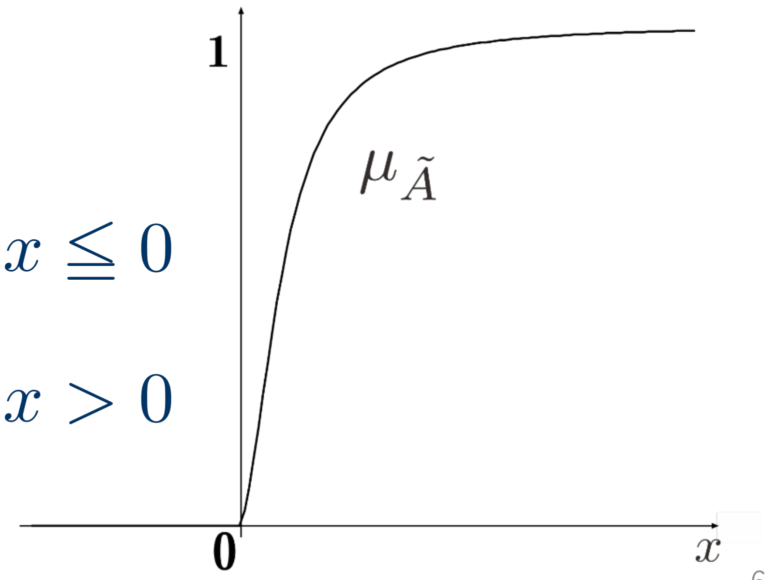
Introduction to Fuzzy Sets (2)

- **Fuzzy Sets**: Fuzzy Sets $\tilde{A}$: Proposed by Lotfi A. Zadeh in 1965
  - A set characterized by a membership function $\mu_{\tilde{A}} : \Omega \Rightarrow [0,1]$ (Ω: Universal set, A set of all objects)
  - For each $x \in \Omega$, $\mu_{\tilde{A}}(x) \in [0,1]$ is assigned
  - The closer to $\left| \begin{matrix} 1 \\ 0 \end{matrix} \right|$ $\mu_{\tilde{A}}(x) \in [0,1]$ is, the $\left| \begin{matrix} \text{higher} \\ \text{lower} \end{matrix} \right|$ the degree of membership of x to $\tilde{A}$.

(Example 2)

Let $\tilde{A}$ be a fuzzy set of real numbers 'much larger than 0'.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leqq 0 \\ \dfrac{1}{1 + \dfrac{100}{x^2}}, & x > 0 \end{cases}$$

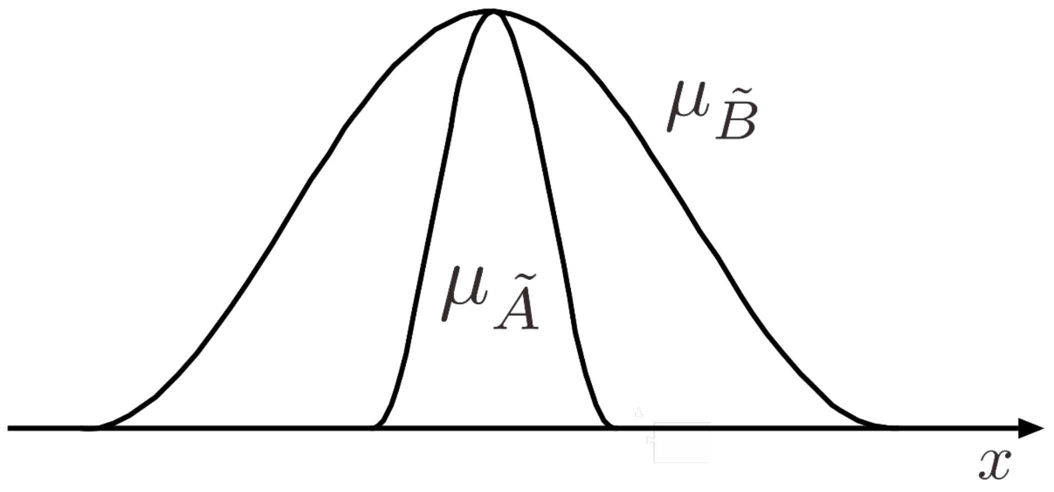


Set Operations of Fuzzy Sets (1)

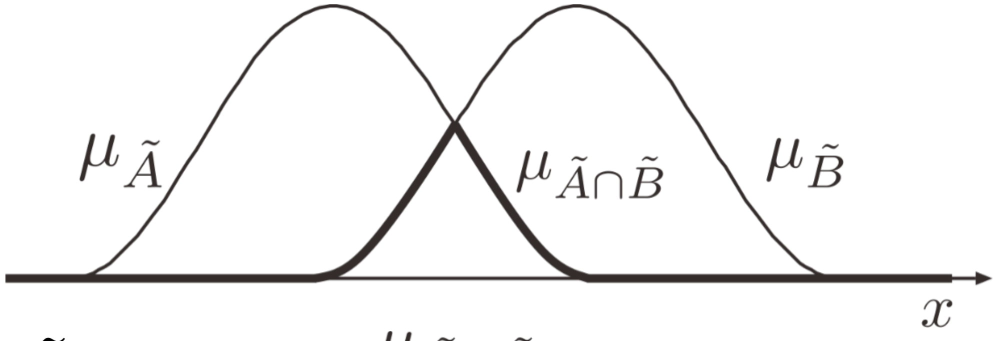
- To make fuzzy sets useful in applications, we need to define calculations of fuzzy sets.

Inclusion relation

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \ \forall x \in \Omega$$



Set Operations (2)



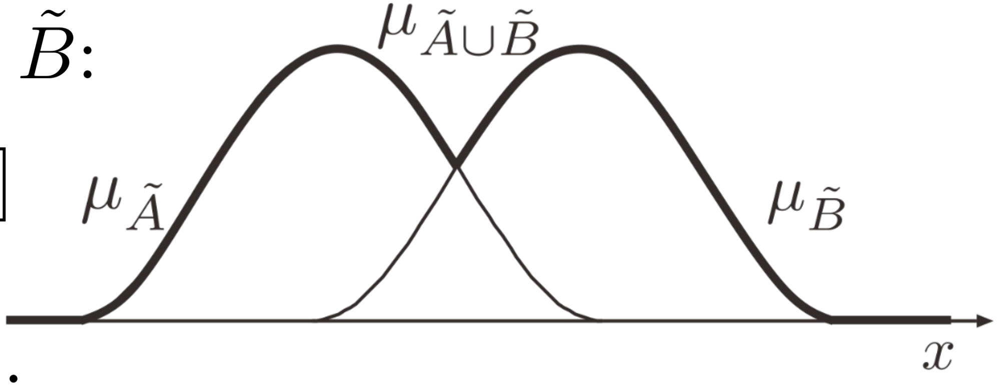
Inclusion relation

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leqq \mu_{\tilde{B}}(x),\ \forall x \in \Omega$$

Intersection $\tilde{A} \cap \tilde{B}$

The maximal set **included** in both $\tilde{A}$ and $\tilde{B}$:

$$\tilde{A} \cap \tilde{B}:\ \mu_{\tilde{A}\cap\tilde{B}}(x) = \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]$$



Union $\tilde{A} \cup \tilde{B}$

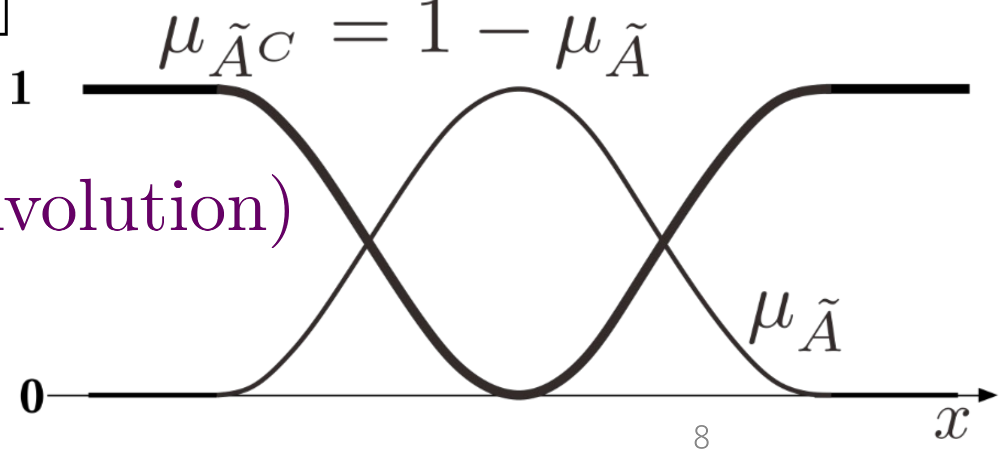
The minimal set **including** both $\tilde{A}$ and $\tilde{B}$:

$$\tilde{A} \cup \tilde{B}:\ \mu_{\tilde{A}\cap\tilde{B}}(x) = \max[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]$$

Complement $\tilde{A}^C$

Independently, we define: $\tilde{A} = (\tilde{A}^C)^C$ (involution)

$$\tilde{A}^C:\ \mu_{\tilde{A}^C}(x) = 1 - \mu_{\tilde{A}}(x)$$



Properties (1)

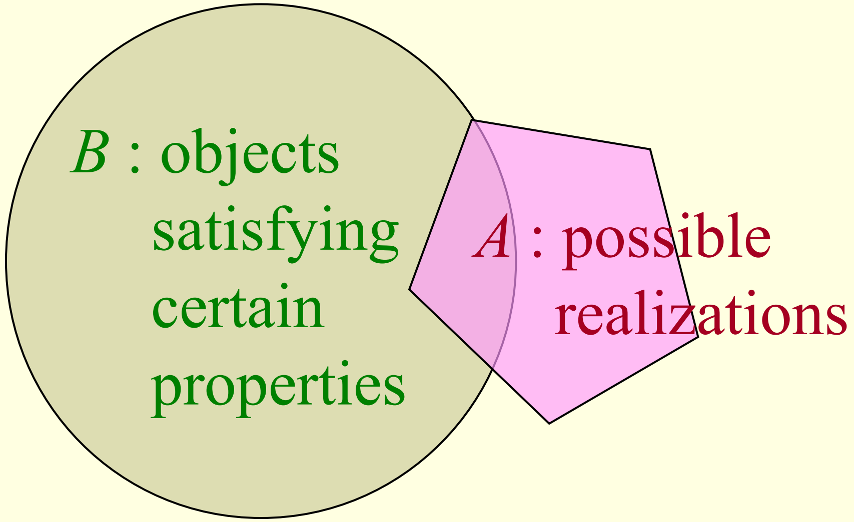
(0)	$\emptyset \subseteq \tilde{A} \subseteq \Omega$
(1)	$\tilde{A} \subseteq \tilde{A}$ (reflexivity)
(2)	$\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$ imply $\tilde{A} = \tilde{B}$ (antisymmetry)
(3)	$\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{C}$ imply $\tilde{A} \subseteq \tilde{C}$ (transitivity)
(4)	$\tilde{A} \cup \tilde{A} = \tilde{A}$ and $\tilde{A} \cap \tilde{A} = \tilde{A}$ (idempotence)
(5)	$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$ and $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$ (commutativity)
(6)	$(\tilde{A} \cup \tilde{B}) \cup \tilde{C} = \tilde{A} \cup (\tilde{B} \cup \tilde{C})$ and $(\tilde{A} \cap \tilde{B}) \cap \tilde{C} = \tilde{A} \cap (\tilde{B} \cap \tilde{C})$ (associativity)
(7)	$\tilde{A} \cup (\tilde{A} \cap \tilde{B}) = \tilde{A}$ and $\tilde{A} \cap (\tilde{A} \cup \tilde{B}) = \tilde{A}$ (absorption)
(8)	$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$ and $\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$ (distributivity)
(9)	$(\tilde{A}^C)^C = \tilde{A}$ (involution)
(10)	$(\tilde{A} \cup \tilde{B})^C = \tilde{A}^C \cap \tilde{B}^C$ and $(\tilde{A} \cap \tilde{B})^C = \tilde{A}^C \cup \tilde{B}^C$ (De Morgan's law)
(11)	$\tilde{A} \cup \Omega = \Omega$, $\tilde{A} \cap \Omega = \tilde{A}$, $\tilde{A} \cup \emptyset = \tilde{A}$ and $\tilde{A} \cap \emptyset = \emptyset$
(12)	Generally, $\tilde{A} \cup \tilde{A}^C \neq \Omega$ and $\tilde{A} \cap \tilde{A}^C \neq \emptyset$ (unsatisfaction of complementary laws)

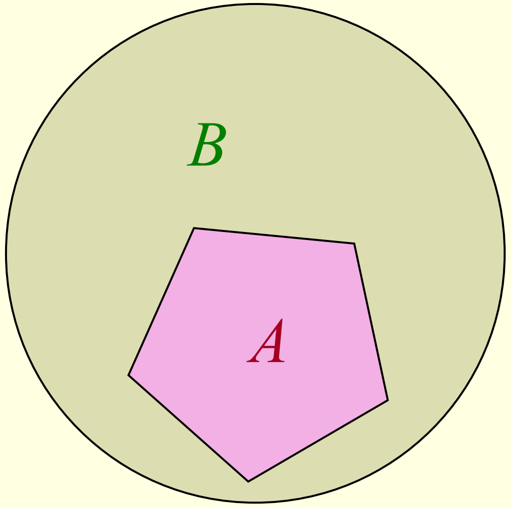
Possibility Theory: Possibility and Necessity

- Basic Treatment: by Possibility and Necessity

  (Common in Non-probabilistic Uncertainty Theories)

Crisp (Non-fuzzy) Case: A: possible region, B: event





Possibility

B is possible $\leftrightarrow B \cap A \neq \emptyset$

$\leftrightarrow \exists z\!: z \in A \wedge z \in B$

Necessity (Certainty)

B is certain $\leftrightarrow B \supseteq A$

$\leftrightarrow \forall z\!: z \in A \rightarrow z \in B$

Possibility Theory: Possibility and Necessity

- Basic Treatment: by Possibility and Necessity

  (Common in Non-probabilistic Uncertainty Theories)

<u>Crisp (Non-fuzzy) Case:</u> A: possible region, B: event

Robustness

sible
izations

Possibility Measure:

$$\Pi_A(B) = \begin{cases} 1, & \text{if } B \cap A = \emptyset \\ 0, & \text{othewise} \end{cases}$$

Necessity Measure:

$$N_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{othewise} \end{cases}$$

Possibility

B is possible $\leftrightarrow$ $B \cap A \neq \emptyset$

$\leftrightarrow$ $\exists z: z \in A \wedge z \in B$

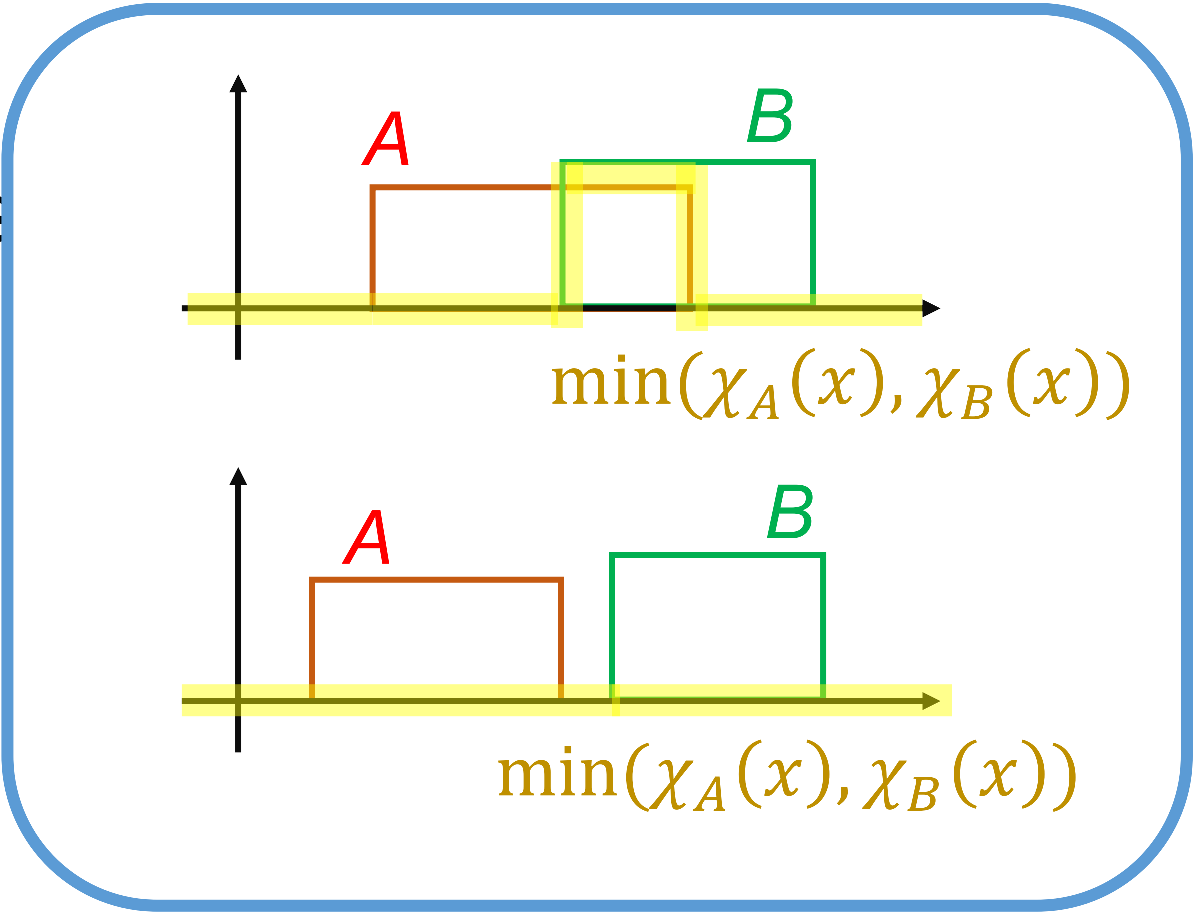
Necessity (Certainty)

B is certain $\leftrightarrow$ $B \supseteq A$

$\leftrightarrow$ $\forall z: z \in A \rightarrow z \in B$

Possibility Theory

- Possibility and Necessity Meas



Possibility Measure

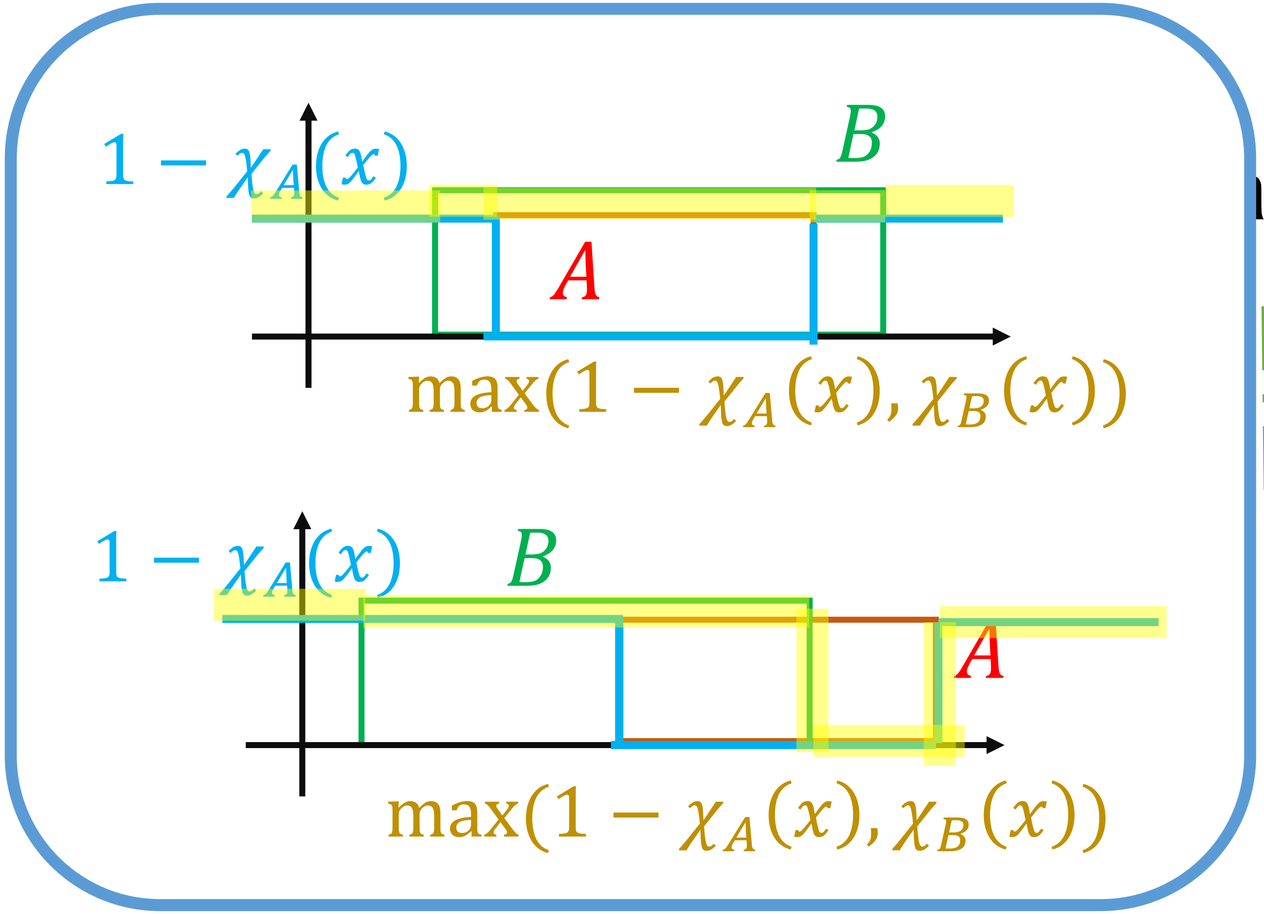
takes 1 if and only if an event is possible and 0 otherwise:

$$\Pi_A(B) = \begin{cases} 1 & : \quad A \cap B \neq \emptyset \\ 0 & : \quad A \cap B = \emptyset \end{cases}$$

Extend these measures to Fuzzy Sets

➔ Express $\Pi_A(B)$ and $N_A(B)$ by the characteristic functions

$$\Pi_A(B) = \sup_x \min(\chi_A(x), \chi_B(x))$$



...sures under Crisp Sets

Necessity Measure

takes 1 if and only if an event is necessary (certain, sure) and 0 otherwise:

$$N_A(B) = \begin{cases} 1 & : \quad A \subseteq B \\ 0 & : \quad A \not\subseteq B \end{cases}$$

Extend these measures to Fuzzy Sets

➔ Express $\Pi_A(B)$ and $N_A(B)$ by the characteristic functions

$$\Pi_A(B) = \sup_x \min(\chi_A(x), \chi_B(x))$$

$$N_A(B) = \inf_x \max(1 - \chi_A(x), \chi_B(x))$$

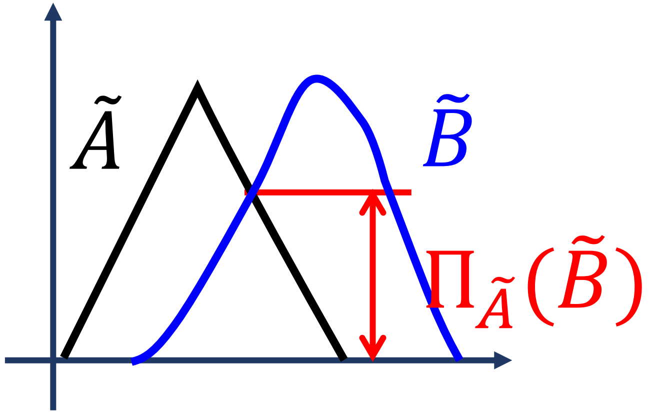
Possibility Theory

- Possibility and Necessity Measures
  - Crisp Case:

$$\Pi_A(B) = \sup_x \min(\chi_A(x), \chi_B(x))$$

$$N_A(B) = \inf_x \max(1 - \chi_A(x), \chi_B(x))$$

  - Fuzzy Case: $\exists x \in \Omega,\ \mu_{\tilde{A}}(x) = 1$ (the normality of $\tilde{A}$) is assumed.

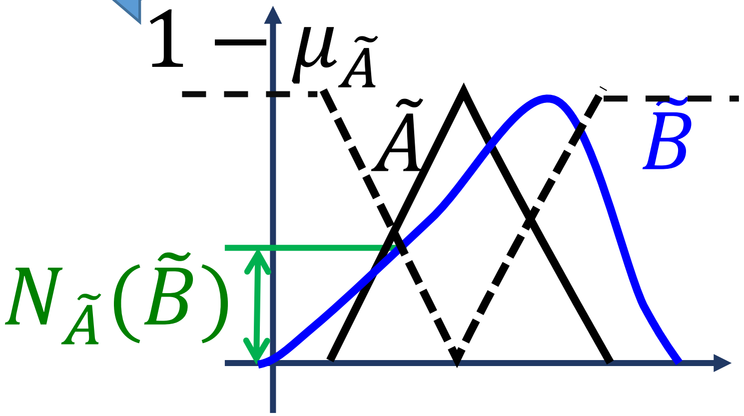


Possibility Measure (Zadeh, 1978)

$$\Pi_{\tilde{A}}(\tilde{B}) = \sup_x \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

Necessity Measure (Dubois & Prade, 1980)

$$N_{\tilde{A}}(\tilde{B}) = \inf_x \max(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$



Possibility Theory

- Properties of possibility and necessity measures

Axioms of possibility measure

1. $\Pi_{\tilde{A}}(\Omega) = 1, \ \Pi_{\tilde{A}}(\emptyset) = 0$
2. $\Pi_{\tilde{A}}(\tilde{B} \cup \tilde{C}) = \max(\Pi_{\tilde{A}}(\tilde{B}), \Pi_{\tilde{A}}(\tilde{C}))$

maxivity

Axioms of necessity measure

1. $N_{\tilde{A}}(\Omega) = 1, \ N_{\tilde{A}}(\emptyset) = 0$
2. $N_{\tilde{A}}(\tilde{B} \cap \tilde{C}) = \min(N_{\tilde{A}}(\tilde{B}), N_{\tilde{A}}(\tilde{C}))$

minivity

Special kinds of Fuzzy Measures

Possibility Theory

- Properties of possibility and necessity measures

Axioms of possibility measure

1. $\Pi_{\tilde{A}}(\Omega) = 1,\ \Pi_{\tilde{A}}(\emptyset) = 0$
2. $\Pi_{\tilde{A}}(\tilde{B} \cup \tilde{C}) = \max(\Pi_{\tilde{A}}(\tilde{B}), \Pi_{\tilde{A}}(\tilde{C}))$

Axioms of necessity measure

1. $N_{\tilde{A}}(\Omega) = 1,\ N_{\tilde{A}}(\emptyset) = 0$
2. $N_{\tilde{A}}(\tilde{B} \cap \tilde{C}) = \min(N_{\tilde{A}}(\tilde{B}), N_{\tilde{A}}(\tilde{C}))$

$$N_{\tilde{A}}(\tilde{B}) = 1 - \Pi_{\tilde{A}}(\tilde{B}^C)$$

$$N_{\tilde{A}}(\tilde{B}) \leq \Pi_{\tilde{A}}(\tilde{B})$$

$$\tilde{B} \subseteq \tilde{C} \Rightarrow \Pi_{\tilde{A}}(\tilde{B}) \leq \Pi_{\tilde{A}}(\tilde{C}),\ N_{\tilde{A}}(\tilde{B}) \leq N_{\tilde{A}}(\tilde{C})$$

$$\max(\Pi_{\tilde{A}}(\tilde{B}), \Pi_{\tilde{A}}(\tilde{B}^C)) \geq 0.5$$

$$\min(N_{\tilde{A}}(\tilde{B}), N_{\tilde{A}}(\tilde{B}^C)) \leq 0.5$$

$$\Pi_{\tilde{A}}(\tilde{B}) > h \Leftrightarrow (\tilde{A})_h \cap (\tilde{B})_h \neq \emptyset$$

$$N_{\tilde{A}}(\tilde{B}) \geq h \Leftrightarrow (\tilde{A})_{1-h} \subseteq [\tilde{B}]_h$$

where $[\tilde{A}]_h = \{x \in \Omega : \mu_{\tilde{A}}(x) \geq h\}$

$(\tilde{A})_h = \{x \in \Omega : \mu_{\tilde{A}}(x) > h\}$

When B is a usual (crisp) set,

$$\max(\Pi_{\tilde{A}}(B), \Pi_{\tilde{A}}(B^C)) = 1$$

$$\min(N_{\tilde{A}}(B), N_{\tilde{A}}(B^C)) = 0$$

$$N_{\tilde{A}}(B) > 0 \Rightarrow \Pi_{\tilde{A}}(B) = 1$$

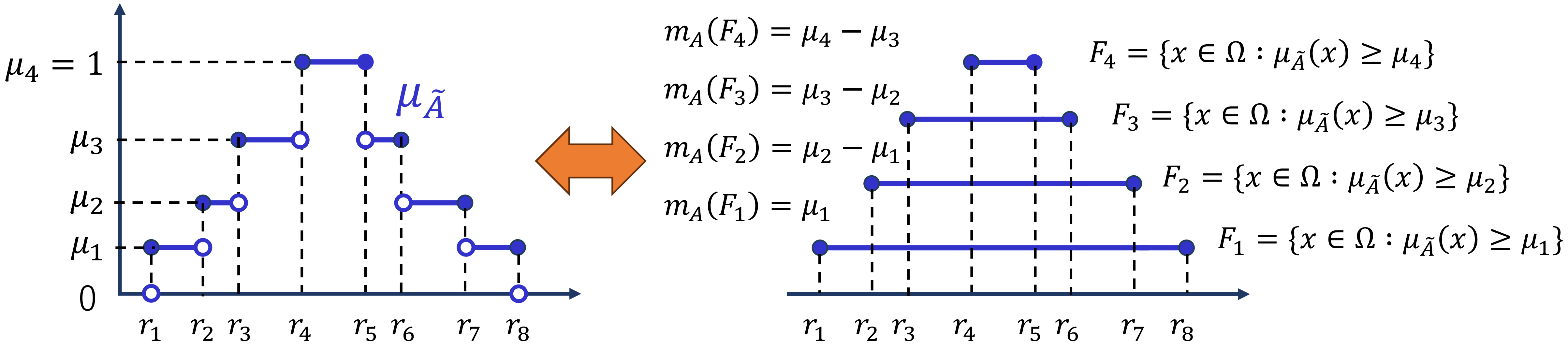
$$\Pi_{\tilde{A}}(B) < 1 \Rightarrow N_{\tilde{A}}(B) = 0$$

Relations to Belief Function

- When B is a crisp set and $\tilde{A}$ is a fuzzy set having discrete membership grades, possibility and necessity measures equal to plausibility and belief functions of a consonant basic probability assignment.

Fuzzy Set $\tilde{A}$

Basic probability assignment m_A



$\mu_{\tilde{A}}(x) = pl(x)$

Consonant Belief Function

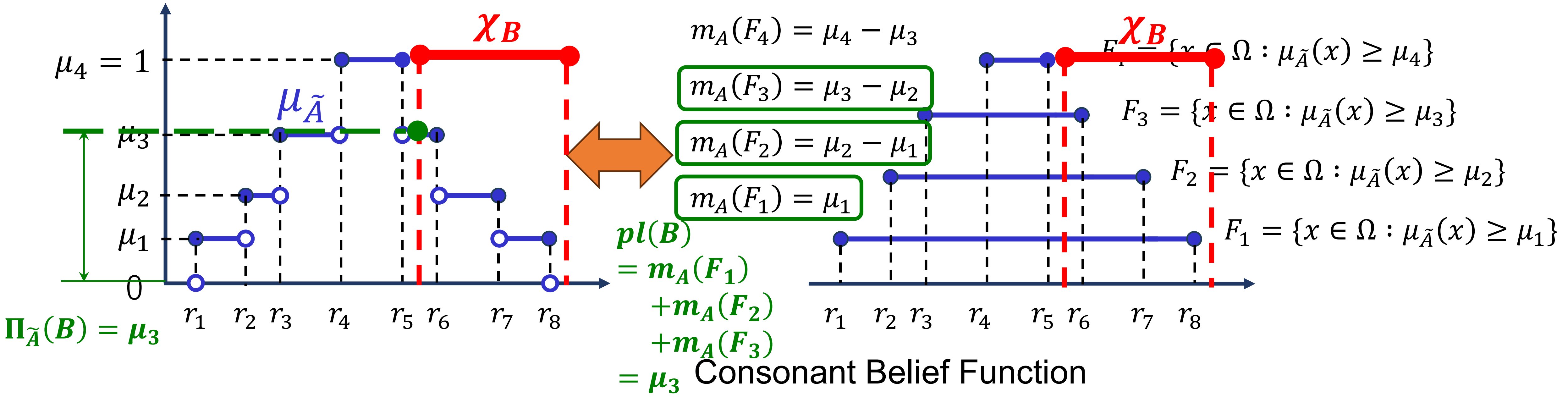
$F_4 \subseteq F_3 \subseteq F_2 \subseteq F_1$: focal elements are nested.

For crisp set B, we obtain

$$\Pi_{\tilde{A}}(B) = \sup_{x} \min\left(\mu_{\tilde{A}}(x), \chi_B(x)\right) = \sup_{x \in B} \mu_{\tilde{A}}(x) = \sum_{F_i : F_i \cap B \neq \emptyset} m_A(F_i) = pl(B)$$

$$N_{\tilde{A}}(B) = \inf_{x} \max(1 - \mu_{\tilde{A}}(x), \chi_B(x)) = \inf_{x \notin B} \left(1 - \mu_{\tilde{A}}(x)\right) = \sum_{F_i : F_i \subseteq B} m_A(F_i) = bel(B)$$

Fuzzy Set $\tilde{A}$ | Basic probability assignment m_A



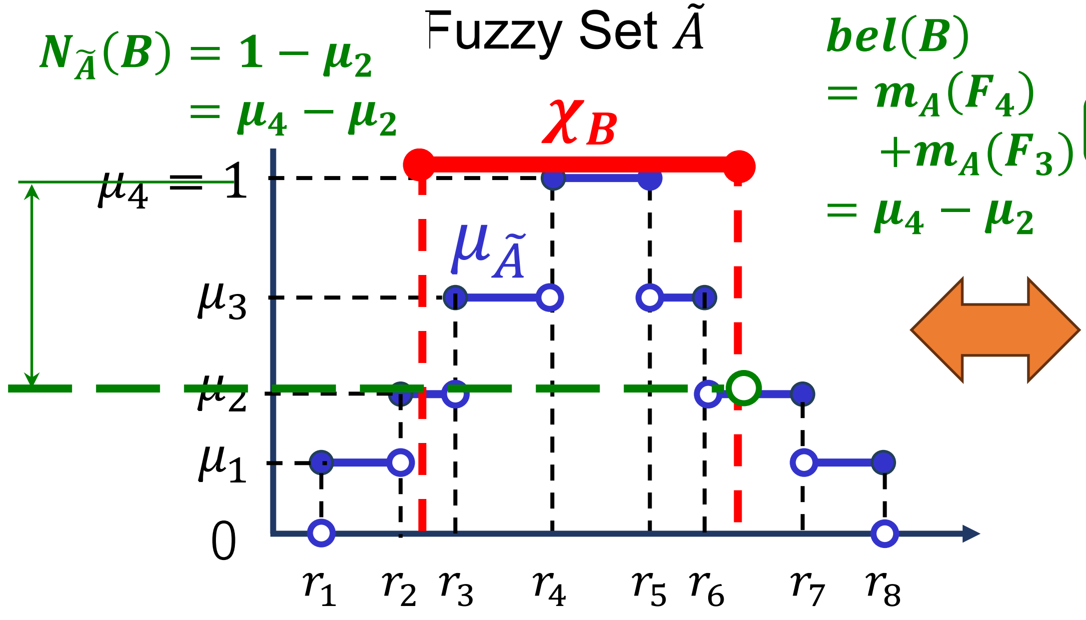
Consonant Belief Function

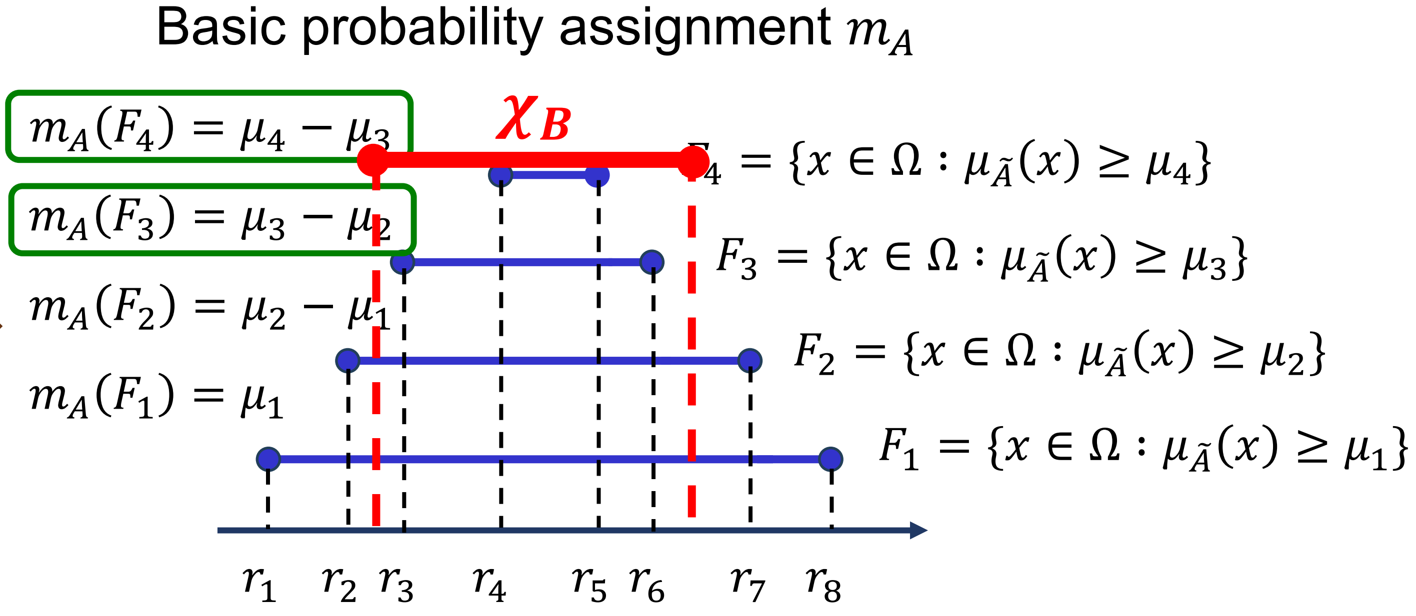
$F_4 \subseteq F_3 \subseteq F_2 \subseteq F_1$: focal elements are nested.

For crisp set B, we obtain

$$\Pi_{\tilde{A}}(B) = \sup_x \min\big(\mu_{\tilde{A}}(x), \chi_B(x)\big) = \sup_{x \in B} \mu_{\tilde{A}}(x) = \sum_{F_i : F_i \cap B \neq \emptyset} m_A(F_i) = pl(B)$$

$$N_{\tilde{A}}(B) = \inf_x \max(1 - \mu_{\tilde{A}}(x), \chi_B(x)) = \inf_{x \notin B} \big(1 - \mu_{\tilde{A}}(x)\big) = \sum_{F_i : F_i \subseteq B} m_A(F_i) = bel(B)$$





Consonant Belief Function

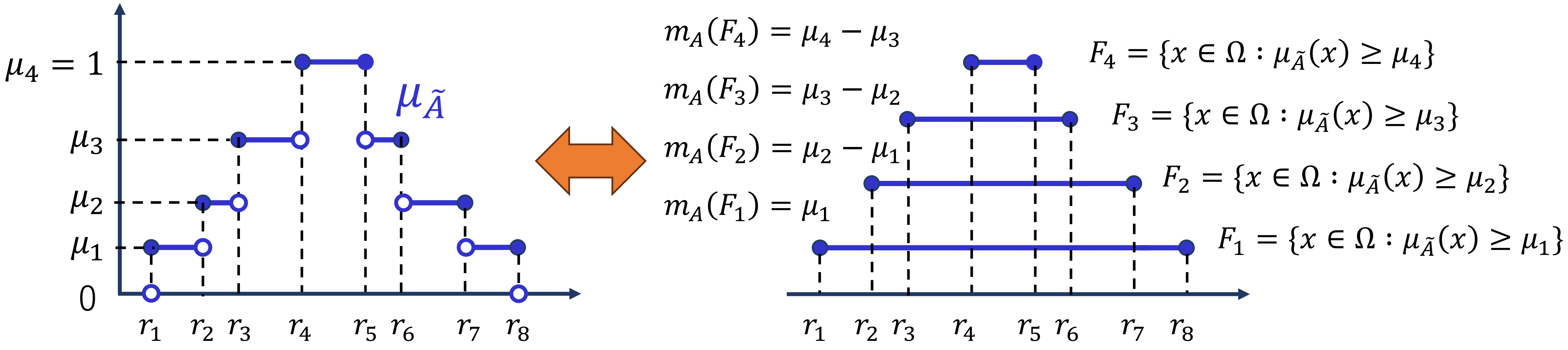
$F_4 \subseteq F_3 \subseteq F_2 \subseteq F_1$: focal elements are nested.

However, **for fuzzy set $\tilde{B}$**, we obtain **Not same !**

$$\Pi_{\tilde{A}}(\tilde{B}) = \sup_x \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right) = (S)\int \mu_{\tilde{B}} \circ \Pi \quad \text{v.s.} \quad (C)\int \mu_{\tilde{B}}\, d(pl)$$

$$N_{\tilde{A}}(\tilde{B}) = \inf_x \max(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = (S)\int \mu_{\tilde{B}} \circ N \quad \text{v.s.} \quad (C)\int \mu_{\tilde{B}}\, d(bel)$$

Fuzzy Set $\tilde{A}$ | Basic probability assignment m_A



Consonant Belief Function

$F_4 \subseteq F_3 \subseteq F_2 \subseteq F_1$: focal elements are nested.

Application to Decision Making ~ Ranking Fuzzy Numbers ~

- Consider a simple decision making problem is to select one from several options whose rewards are estimated by fuzzy numbers.

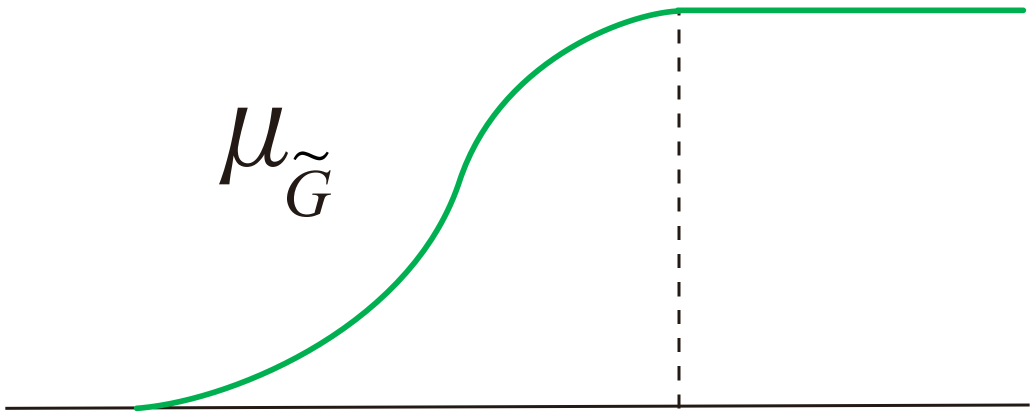
alternative (option)	expected income (reward)
o_1	$\tilde{A}_1$
o_2	$\tilde{A}_2$
$\vdots$	$\vdots$
o_n	$\tilde{A}_n$

Ranking alternatives using a fuzzy goal

- Fuzzy goal (fuzzy set of satisfactory rewards)

We suppose that the decision maker can specify a fuzzy goal $\tilde{G}$. The membership grade $\mu_{\tilde{G}}(x)$ of the fuzzy goal $\tilde{G}$ shows the degree of satisfaction.

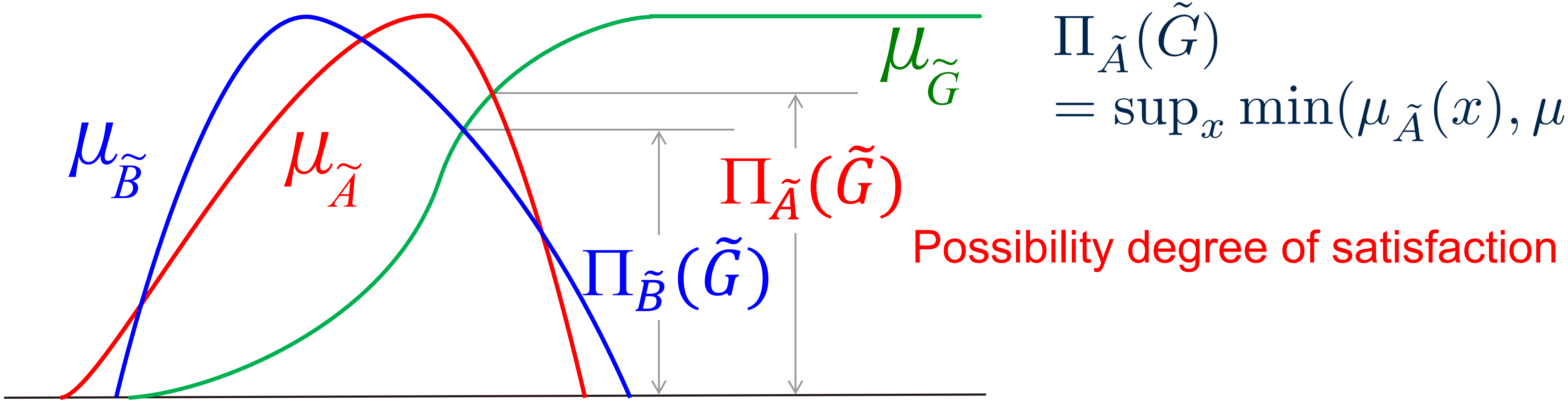
$\mu_{\tilde{G}}$ is similar to a utility function



Ranking alternatives using a fuzzy goal

- Possibility measure maximization

A principle,
"the higher the possibility of satisfaction, the better the solution".



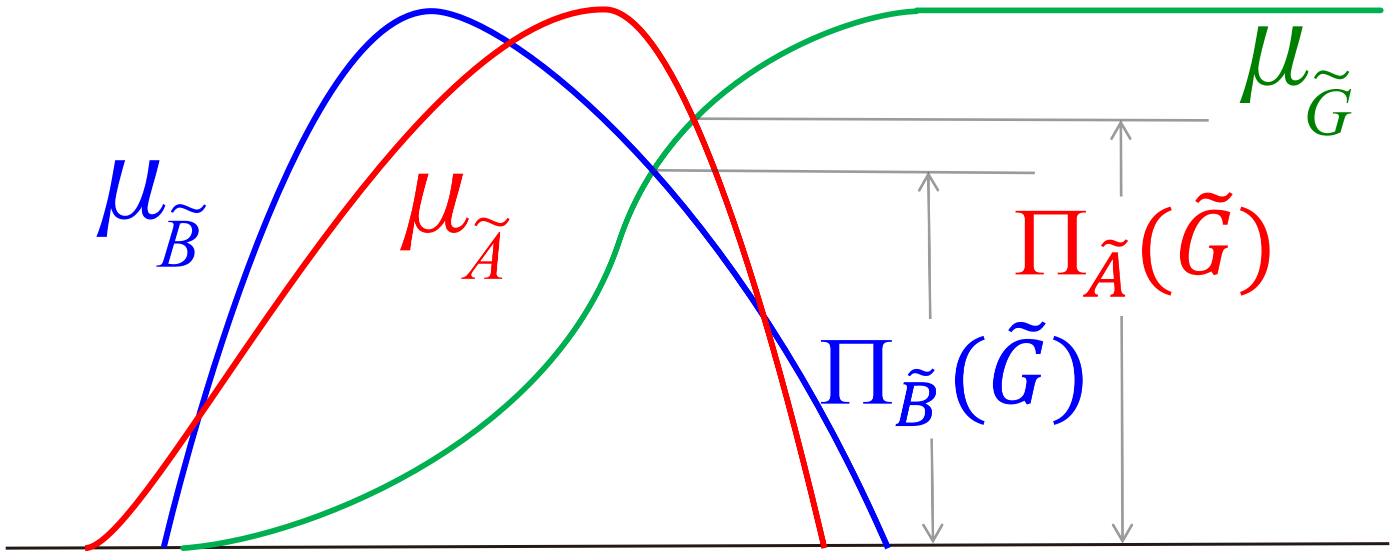
$$\Pi_{\tilde{A}}(\tilde{G}) = \sup_x \min(\mu_{\tilde{A}}(x), \mu_{\tilde{G}}(x))$$

Ranking alternatives using a fuzzy goal

- Possibility measure maximization

Possibility measure maximization: Select $\tilde{A}_{i^*}$ such that

$$\Pi_{\tilde{A}_{i^*}}(\tilde{G}) = \max_i \Pi_{\tilde{A}_i}(\tilde{G})$$



$$\Pi_{\tilde{A}}(\tilde{G}) = \sup_x \min(\mu_{\tilde{A}}(x), \mu_{\tilde{G}}(x))$$

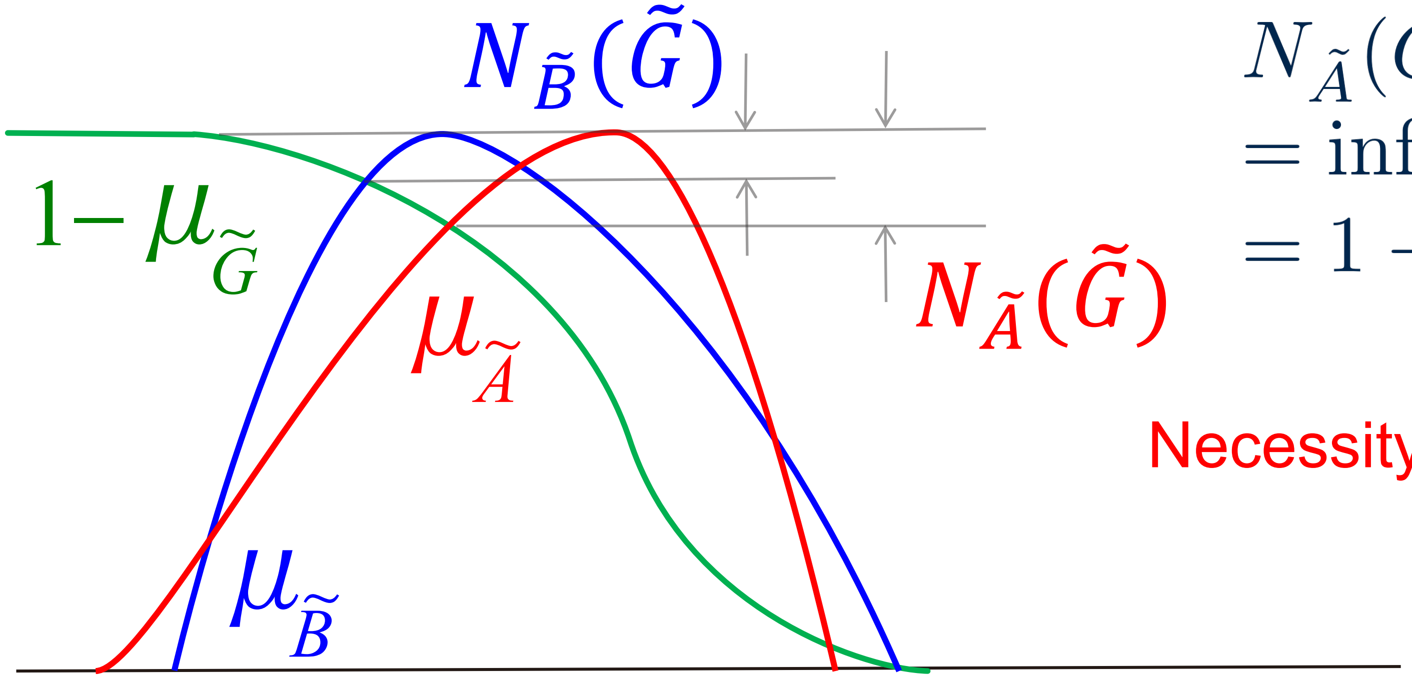
$$= (S)\int \mu_{\tilde{G}} \circ \Pi$$

Corresponding to Expected Utility

Ranking alternatives using a fuzzy goal

- Necessity measure maximization

A principle, "the higher the necessity (certainty) of satisfaction, the better the solution".



$$N_{\tilde{A}}(\tilde{G})$$
$$= \inf_x \max(1-\mu_{\tilde{A}}(x), \mu_{\tilde{G}}(x))$$
$$= 1-\sup_x \min(\mu_{\tilde{A}}(x), 1-\mu_{\tilde{G}}(x))$$

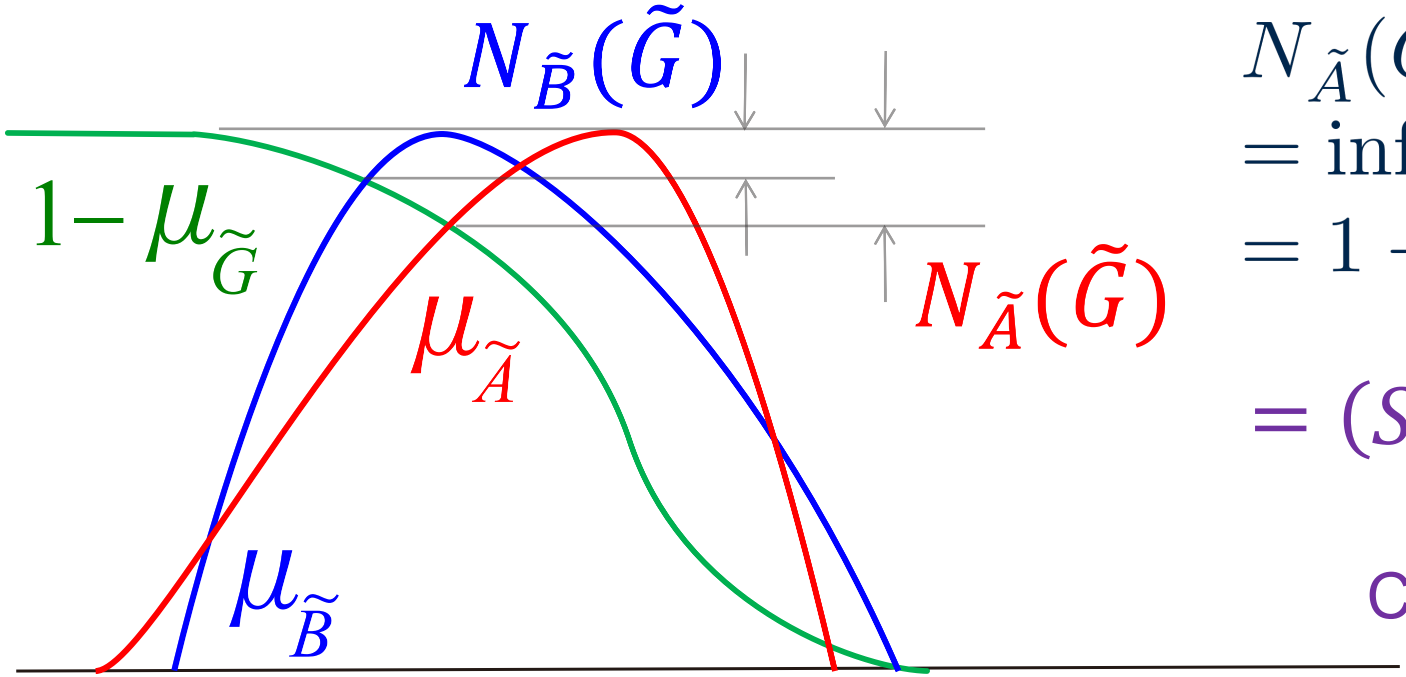
Necessity degree of satisfaction

Ranking alternatives using a fuzzy goal

- Necessity measure maximization

Necessity measure maximization: Select $\tilde{A}_{i*}$ such that

$$N_{\tilde{A}_{i*}}(\tilde{G}) = \max_i N_{\tilde{A}_i}(\tilde{G})$$



$$N_{\tilde{A}}(\tilde{G})$$
$$= \inf_x \max(1-\mu_{\tilde{A}}(x), \mu_{\tilde{G}}(x))$$
$$= 1 - \sup_x \min(\mu_{\tilde{A}}(x), 1-\mu_{\tilde{G}}(x))$$
$$= (S)\int \mu_{\tilde{G}} \circ N$$

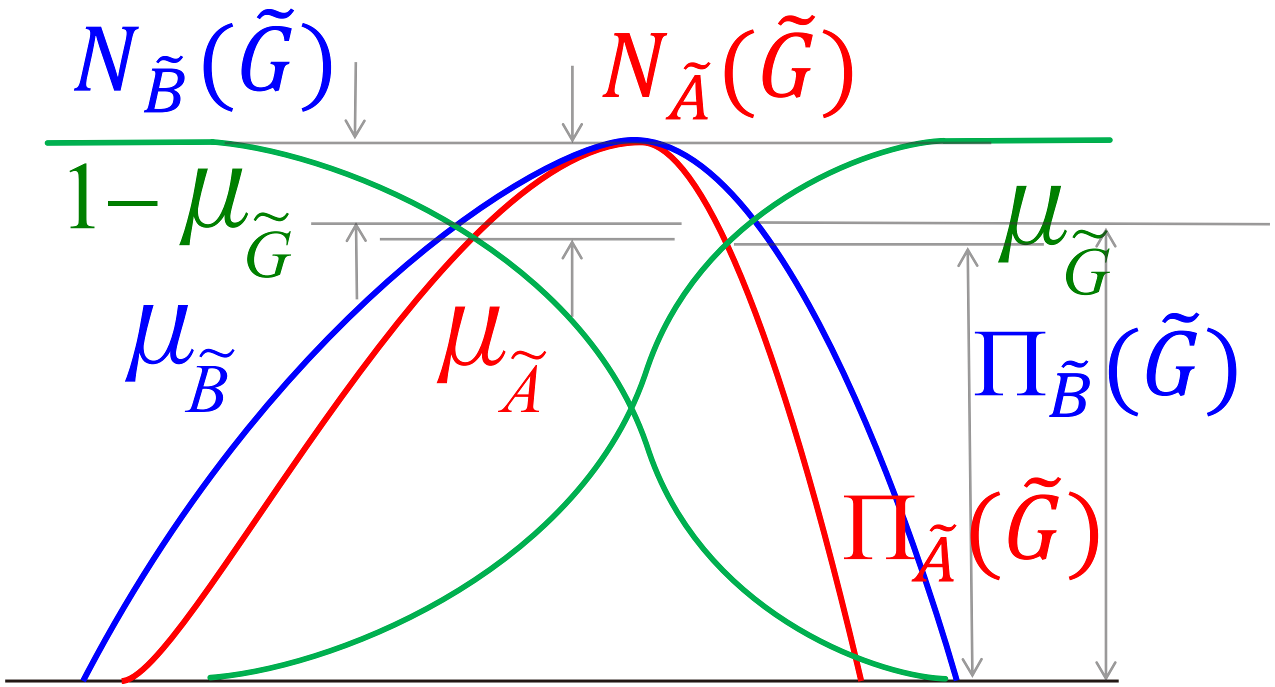
Corresponding to Expected Utility

Ranking alternatives using a fuzzy goal

Inuiguchi & Ichihashi (1990)

- Decision-maker's attitude toward uncertainty

Definition
Uncertainty averse: $\tilde{A} \subseteq \tilde{B} \Rightarrow \tilde{A} \succeq \tilde{B}$
Uncertainty prone: $\tilde{A} \subseteq \tilde{B} \Rightarrow \tilde{B} \succeq \tilde{A}$
Uncertainty neutral: Neither uncertainty averse nor uncertainty prone.



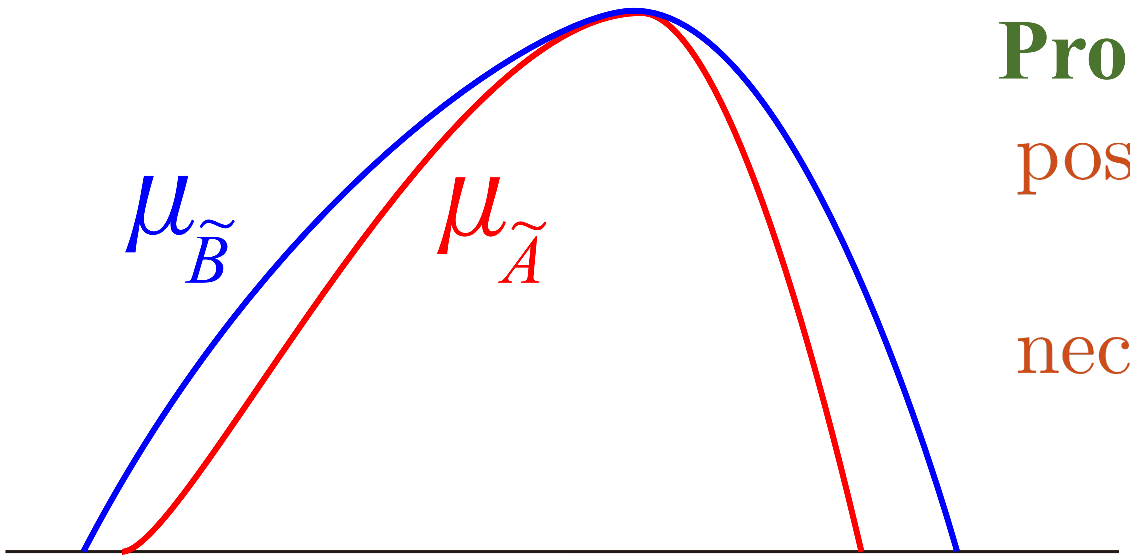
Ranking alternatives using a fuzzy goal

- Decision-maker's attitude toward uncertainty

Definition Uncertainty averse: $\tilde{A} \subseteq \tilde{B} \Rightarrow \tilde{A} \succeq \tilde{B}$

Uncertainty prone: $\tilde{A} \subseteq \tilde{B} \Rightarrow \tilde{B} \succeq \tilde{A}$

Uncertainty neutral: Neither uncertainty averse nor uncertainty prone.



Properties:

possibility measure maximization $\Rightarrow$ uncertainty prone

necessity measure maximization $\Rightarrow$ uncertainty averse

Ranking alternatives using a fuzzy goal

- Relative possibility measure maximization

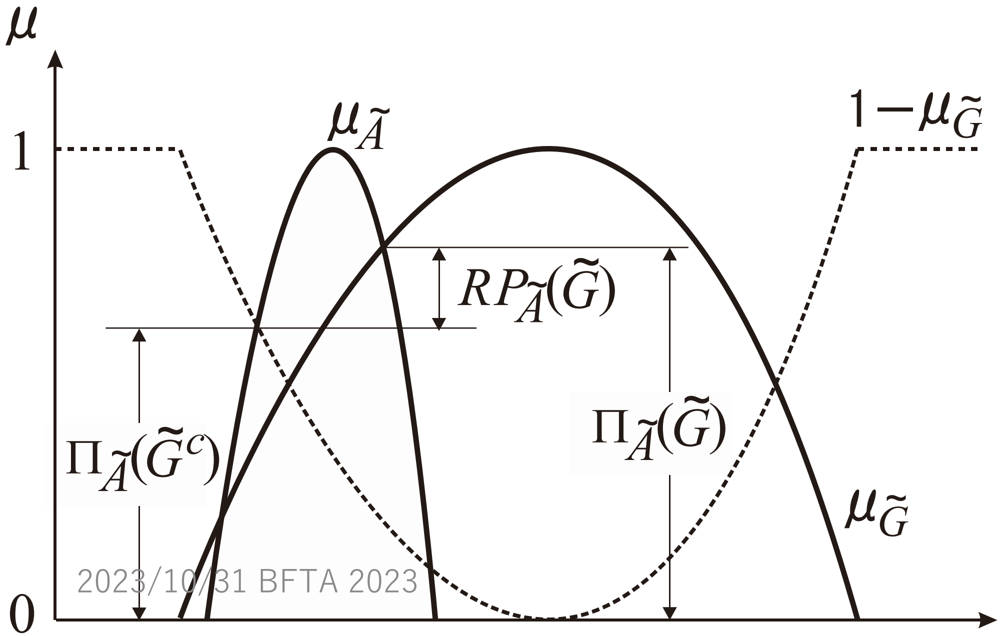
Inuiguchi & Ichihashi (1990)

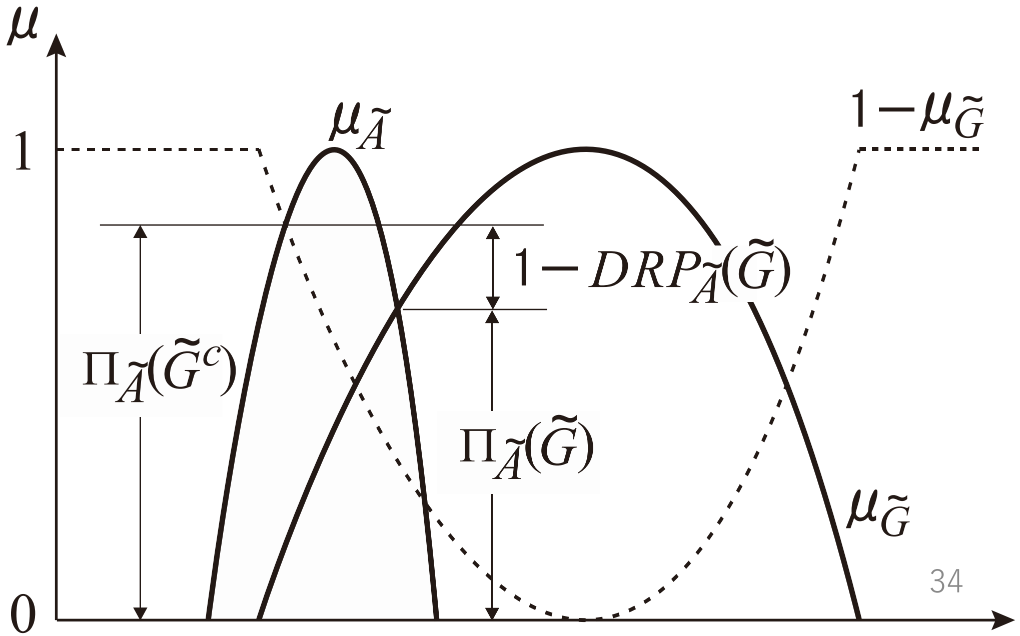
Relative possibility measure (RP): The degree to what extent the possibility of satisfaction is larger than the possibility of unsatisfaction.

$$RP_{\tilde{A}}(\tilde{G}) = \max(\Pi_{\tilde{A}}(\tilde{G}) - \Pi_{\tilde{A}}(\tilde{G}^c), 0).$$

Dual RP measure (DRP): The degree to what extent the possibility of satisfaction is not smaller than the possibility of unsatisfaction.

$$DRP_{\tilde{A}}(\tilde{G}) = \min(1 - \Pi_{\tilde{A}}(\tilde{G}^c) + \Pi_{\tilde{A}}(\tilde{G}), 1).$$





Ranking alternatives using a fuzzy goal

- Relative possibility measure maximization

We have the following relation:

One of $RP_{\tilde{A}}(\tilde{G})$ and $DRP_{\tilde{A}}(\tilde{G})$ is constant.

$$RP_{\tilde{A}}(\tilde{G}) > 0 \;\Rightarrow\; DRP_{\tilde{A}}(\tilde{G}) = 1$$
$$DRP_{\tilde{A}}(\tilde{G}) < 1 \;\Rightarrow\; RP_{\tilde{A}}(\tilde{G}) = 0$$

$$RP_{\tilde{A}}(\tilde{G}) + DRP_{\tilde{A}}(\tilde{G}) = \Pi_{\tilde{A}}(\tilde{G}) + N_{\tilde{A}}(\tilde{G})$$
$$\Rightarrow \left[\Pi_{\tilde{A}}(\tilde{G}) + N_{\tilde{A}}(\tilde{G}) \geq \Pi_{\tilde{B}}(\tilde{G}) + N_{\tilde{B}}(\tilde{G}) \;\Leftrightarrow\; \tilde{A} \succeq^{RP} \tilde{B}\right]$$

Relative possibility measure maximization:

$$\max_i \left(\Pi_{\tilde{A}_i}(\tilde{G}) + N_{\tilde{A}_i}(\tilde{G})\right)$$

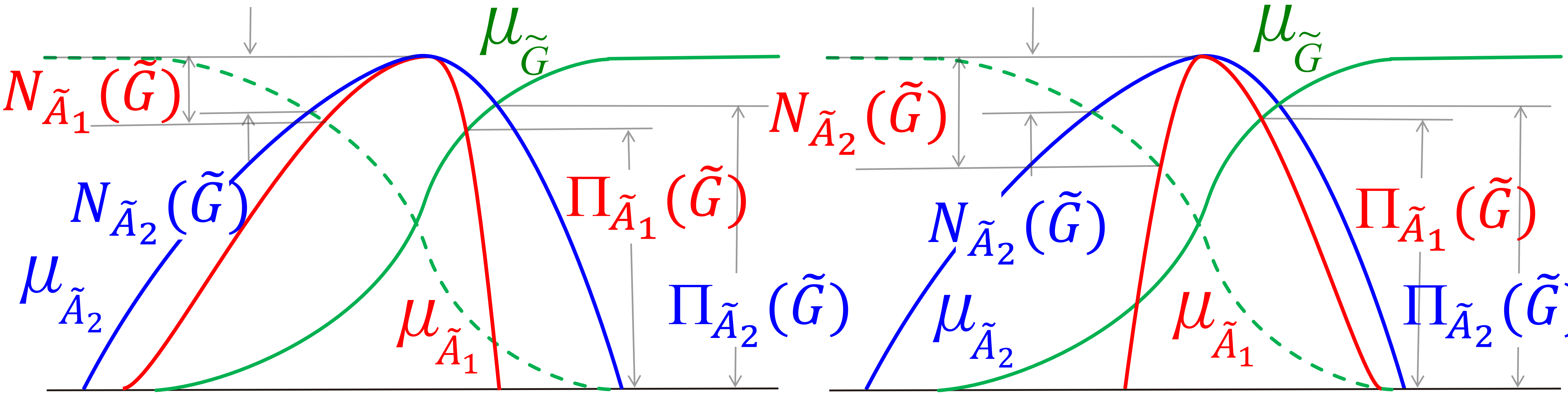
Ranking alternatives using a fuzzy goal

- Relative possibility measure maximization

Relative possibility measure maximization:

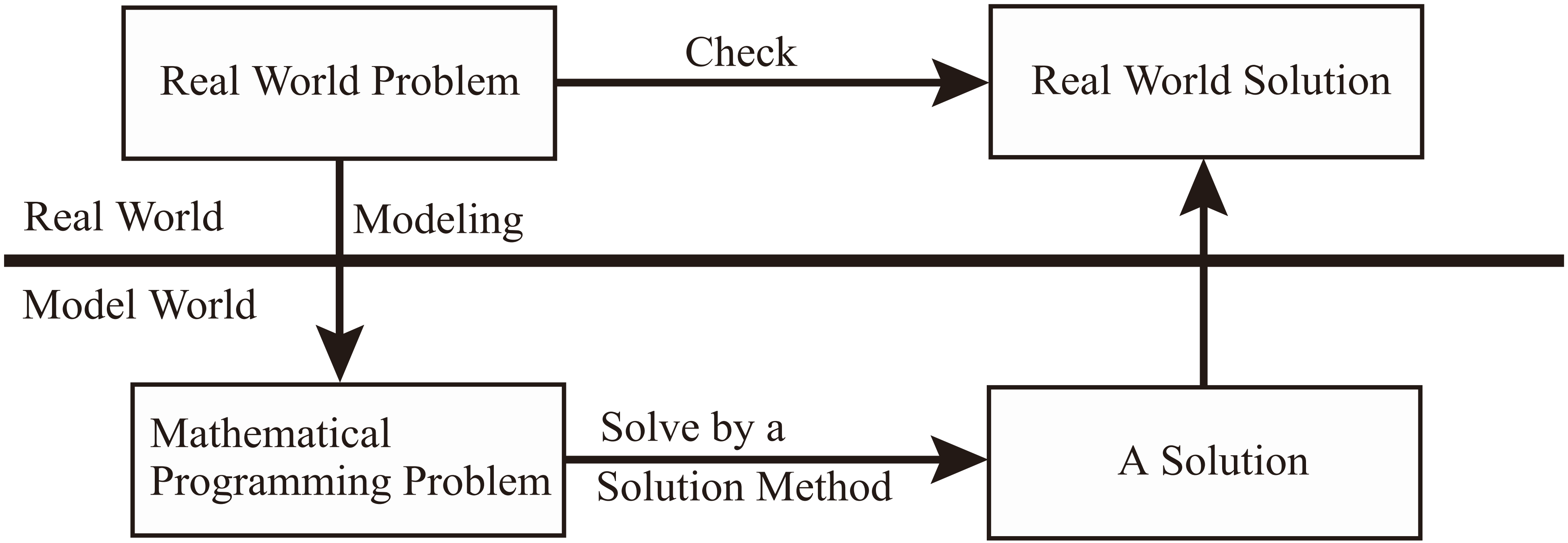
$$\max_i \left(\Pi_{\tilde{A}_i}(\tilde{G}) + N_{\tilde{A}_i}(\tilde{G}) \right)$$

Relative possibility measure maximization $\Rightarrow$ uncertainty neutral



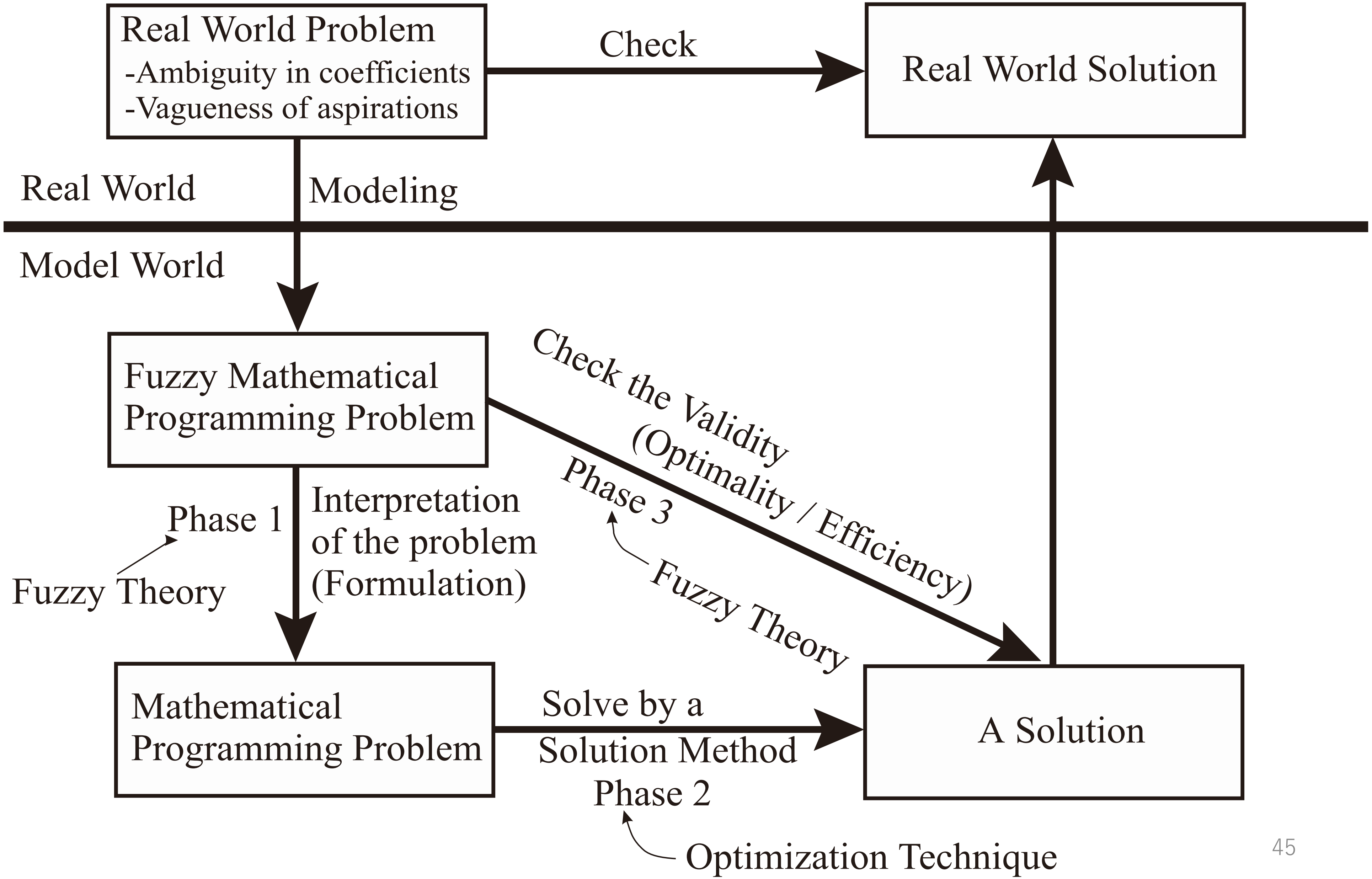
Fuzzy Mathematical Programming Approach

- Conventional MP approach



Fuzzy Mathematical Programming Approach

- Fuzzy MP approach



How to Use Fuzzy (Possibilistic) Programming ?

Inuiguchi & Ramik (2000)

- Production Planning

In a factory, the factory manager intends to manufacture a new product A. The total manufacturing process is composed of three processes, Process 1, Process 2 and Process 3. This is the same as that of Product B. The estimated processing time for manufacturing a batch of Product A at each process is as follows: about 2 time units at Process 1, about 4 time units at Process 2 and about 1 time unit at Process 3. On the other hand, the processing time for manufacturing a batch of Product B at each process is as follows: about 3 time units at Process 1, about 2 time units at Process 2 and about 3 time units at Process 3. The working time at Process 1 is restricted by 240 time units, that at Process 2 is restricted by 400 time units and that at Process 3 is restricted by 210 time units. The profit rates (100$/batch) of Products A and B are about 5 and about 7, respectively. How many Products A and B should be manufactured in order to maximize the total profit ?

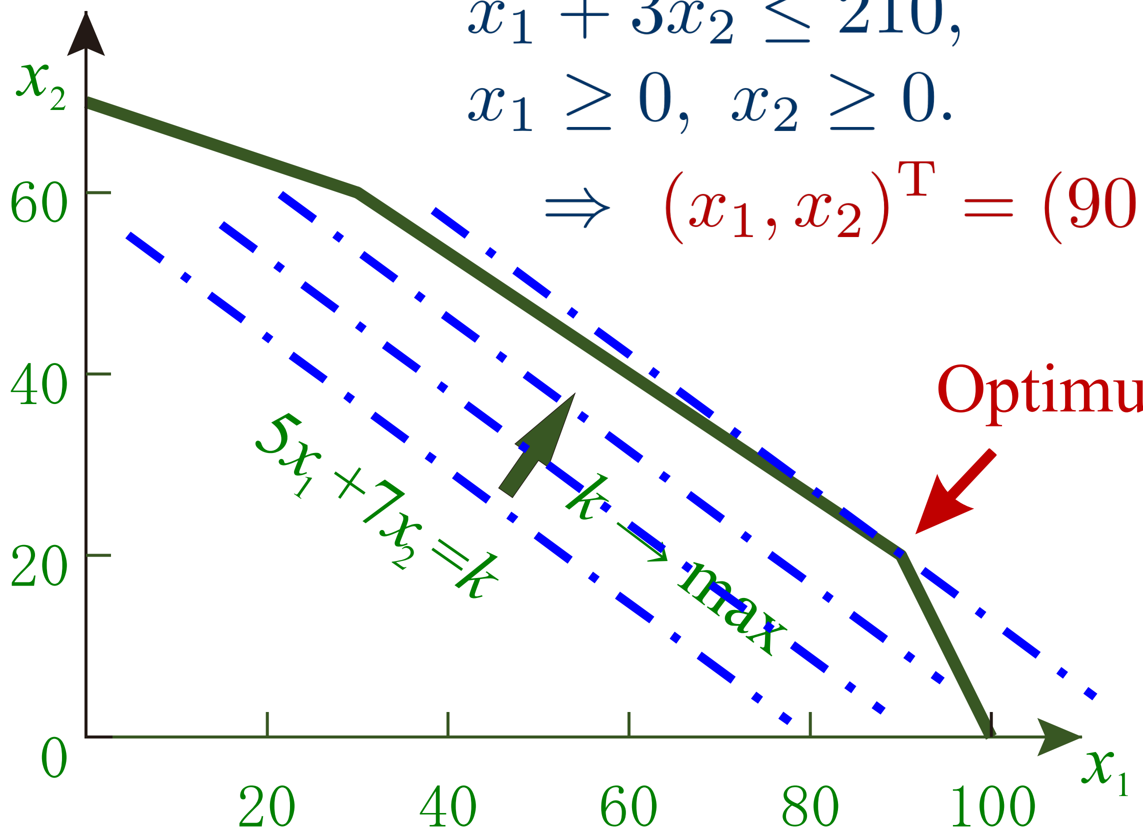
How to Use Fuzzy (Possibilistic) Programming ?

- The conventional linear programming approach

	New Product A: x_1	Product B: x_2	Working Time
Process 1	$\tilde{2}$	$\tilde{3}$	$\leqq$ 240
Process 2	$\tilde{4}$	$\tilde{2}$	$\leqq$ 400
Process 3	$\tilde{1}$	$\tilde{3}$	$\leqq$ 210
Profit	$\tilde{5}$	$\tilde{7}$	max

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 7x_2, \\ \text{sub. to} \quad & 2x_1 + 3x_2 \leq 240, \\ & 4x_1 + 2x_2 \leq 400, \\ & x_1 + 3x_2 \leq 210, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

$$\Rightarrow \ (x_1, x_2)^{\mathrm{T}} = (90, 20)^{\mathrm{T}}$$



How to Use Fuzzy (Possibilistic) Programming ?

- The conventional linear programming approach

$$(x_1, x_2)^{\mathrm{T}} = (90, 20)^{\mathrm{T}}$$

$$2 \times 90 + 3 \times 20 = 240$$
$$4 \times 90 + 2 \times 20 = 400$$
$$90 + 3 \times 20 < 210$$

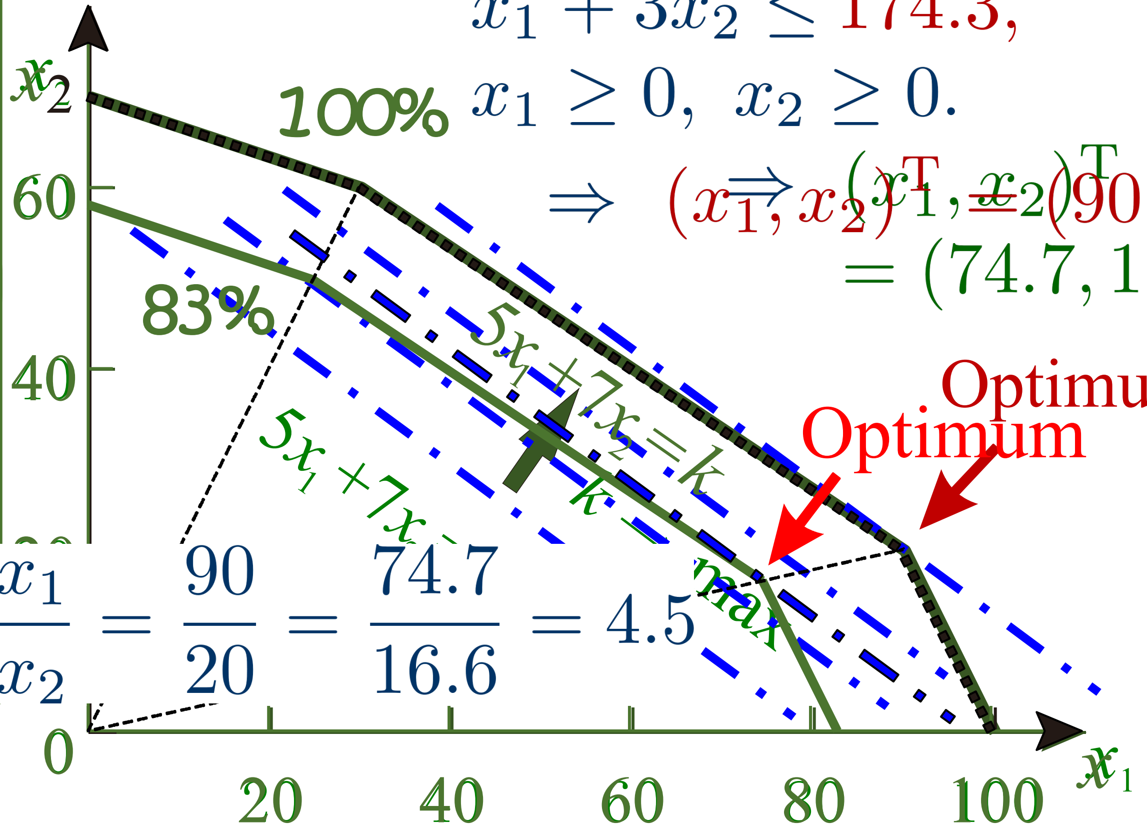
Risky in the sense of the Feasibility !!

83%

$$240 \rightarrow 199.2$$
$$400 \rightarrow 332$$
$$210 \rightarrow 174.3$$

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 7x_2, \\ \text{sub. to} \quad & 2x_1 + 3x_2 \le 199.2, \\ & 4x_1 + 2x_2 \le 332, \\ & x_1 + 3x_2 \le 174.3, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{aligned}$$

$$\Rightarrow (x_1, x_2)^{\mathrm{T}} = (90, 20)^{\mathrm{T}}$$
$$= (74.7, 16.6)^{\mathrm{T}}$$



$$\frac{x_1}{x_2} = \frac{90}{20} = \frac{74.7}{16.6} = 4.5$$

1

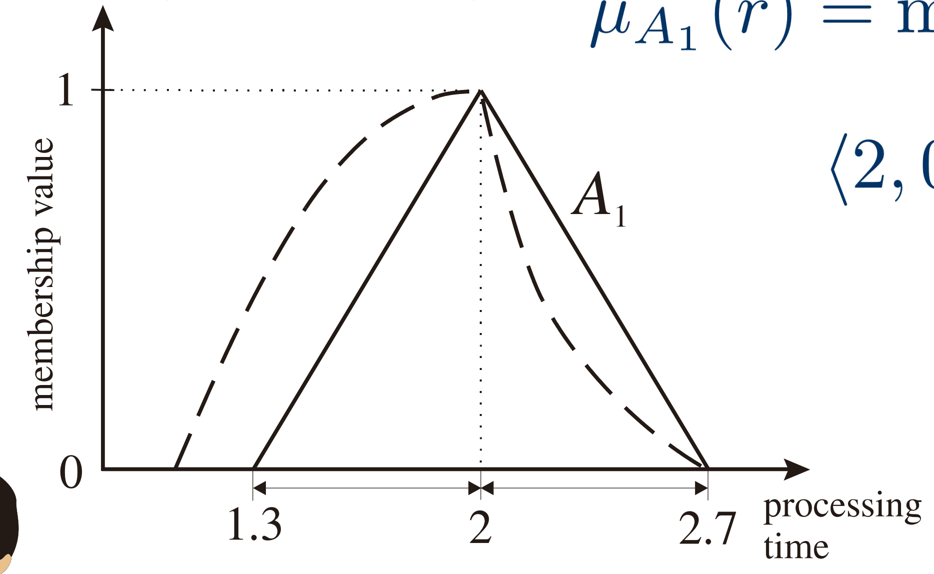
How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach

The processing time: $\Longrightarrow$ Fuzzy Number:
'about 2 time units' (Interview)

$$\mu_{A_1}(r) = \max\left(0, 1 - \frac{10|r-2|}{7}\right)$$

$$\langle 2, 0.7 \rangle$$

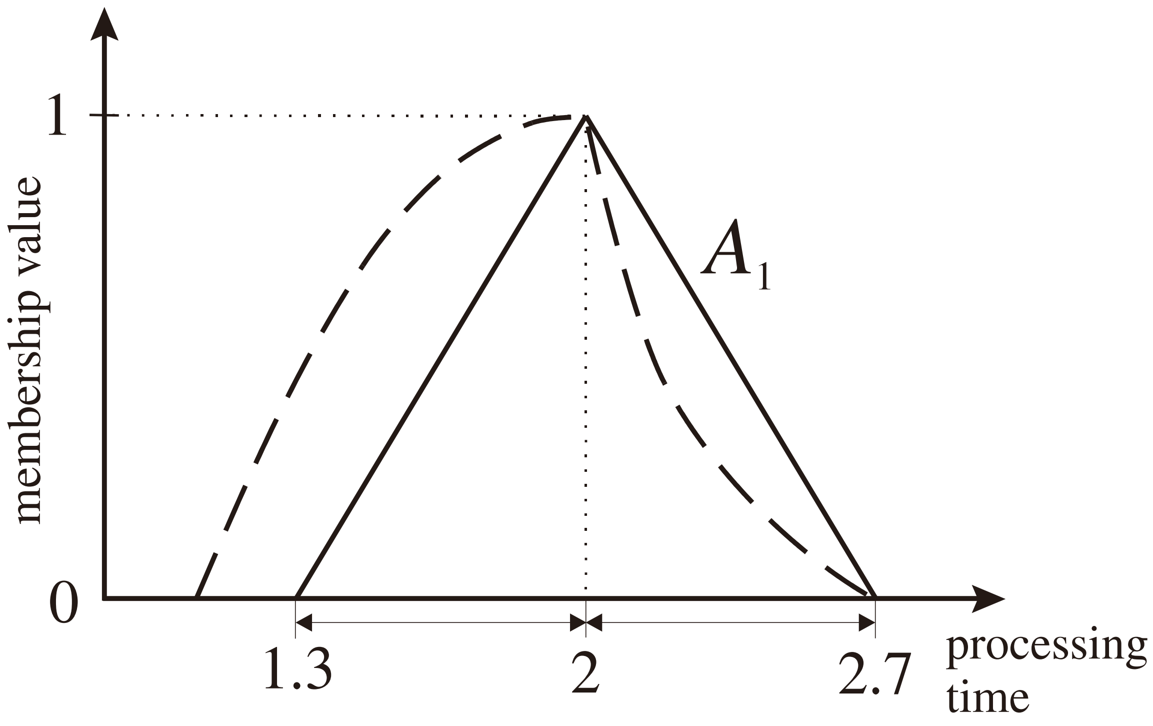


Ask crisp values

Accept fuzzy values

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Meaning of a Symmetric Triangular Fuzzy Number



Symmetric Triangular Fuzzy Number $\langle 2, 0.7 \rangle$

- '2' is the most plausible value.
- At most 2.7, i.e., more than 2.7 is impossible.
- At least 1.3, i.e., less than 1.3 is impossible.
- Possibility more than 2 and less than 2 are the same.
- The membership value linearly decreases as it departs from 2.

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach

	New Product A: x_1	Product B: x_2	Working Time
Process 1	$\tilde{2}$	$\tilde{3}$	$\leqq$ 240
Process 2	$\tilde{4}$	$\tilde{2}$	$\leqq$ 400
Process 3	$\tilde{1}$	$\tilde{3}$	$\leqq$ 210
Profit	$\tilde{5}$	$\tilde{7}$	max

The obtained fuzzy numbers

product	A	B	Working Time
Process 1	$\tilde{A}_1 = \langle 2, 0.7 \rangle$	$\tilde{B}_1 = \langle 3, 0.5 \rangle$	240
Process 2	$\tilde{A}_2 = \langle 4, 1.5 \rangle$	$\tilde{B}_2 = \langle 2, 0.3 \rangle$	400
Process 3	$\tilde{A}_3 = \langle 1, 0.5 \rangle$	$\tilde{B}_3 = \langle 3, 0.3 \rangle$	210
profit rate	$\tilde{C}_1 = \langle 5, 1 \rangle$	$\tilde{C}_2 = \langle 7, 0.7 \rangle$	

A is more uncertain than B.

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach

$$
\begin{aligned}
\text{maximize} \quad & c_1x_1 + c_2x_2, \\
\text{subject to} \quad & a_1x_1 + b_1x_2 \leq 240, \\
& a_2x_1 + b_2x_2 \leq 400, \\
& a_3x_1 + b_3x_2 \leq 210, \\
& x_1 \geq 0, \ x_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad & \text{possibilistic variable } a_i \quad && \text{restricted by } \tilde{A}_i \\
& \text{possibilistic variable } b_i && \text{restricted by } \tilde{B}_i \\
& \text{possibilistic variable } c_i && \text{restricted by } \tilde{C}_i
\end{aligned}
$$

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Calculation of Possibilistic Linear Function value (Extension Principle)

$f_0(x_1, x_2) = c_1 x_1 + c_2 x_2$ is restricted by a fuzzy number $F_0(x_1, x_2)$;

$$\mu_{F_0(x_1,x_2)}(r) = \sup_{\substack{p,q \\ r=px_1+qx_2}} \min(\mu_{\tilde{C}_1}(p), \mu_{\tilde{C}_2}(q))$$

Example: $z = \sum_{j=1}^{n} k_j y_j, \ \ y_i \in Y_j = \langle y_j^{\mathrm{c}}, w_j \rangle$

$$\cancel{k_i \geq 0, \ \ \forall i} \Rightarrow \ \ z \in Z = \left\langle \sum_{j=1}^{n} k_j y_j^{\mathrm{c}}, \sum_{j=1}^{n} |k_j| w_j \right\rangle$$

Fuzzy linear function value with triangular fuzzy numbers

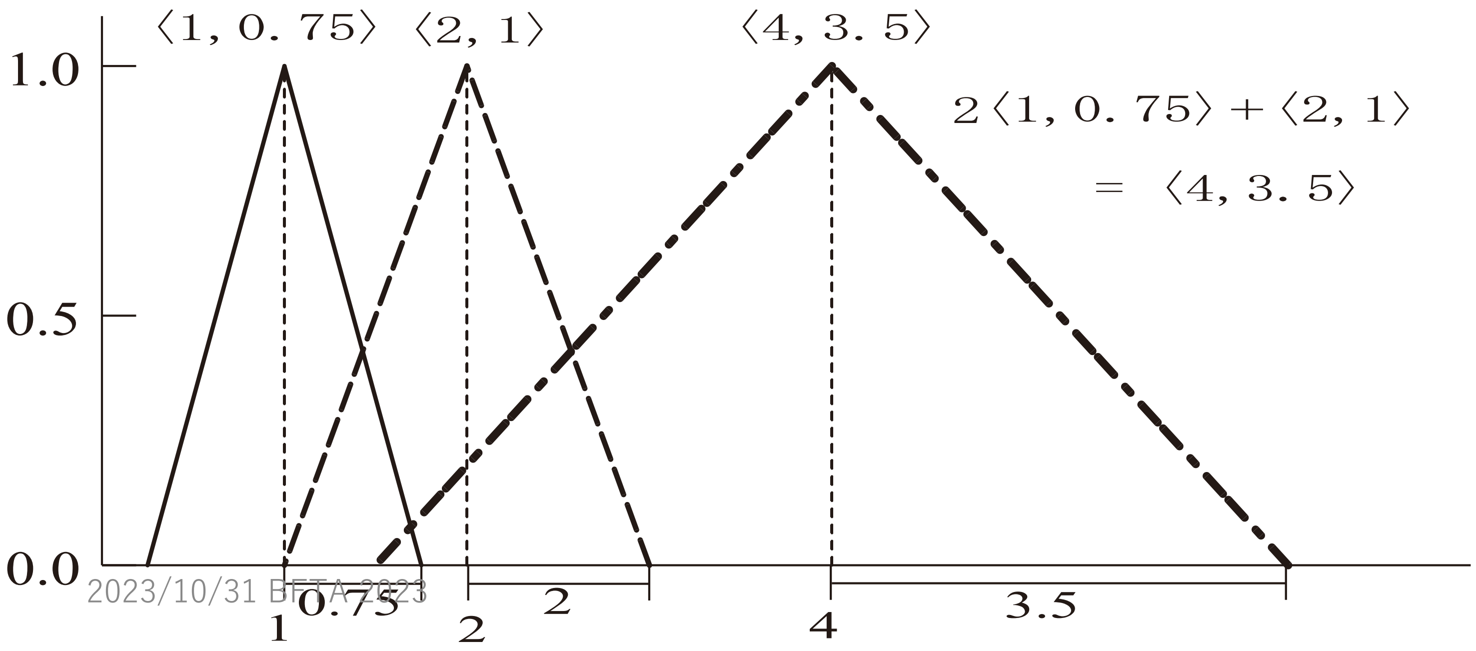
➔ A triangular fuzzy number

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Calculation of Possibilistic Linear Function value (Extension Principle)

Example: $$z = \sum_{j=1}^{n} k_j y_j, \quad y_i \in Y_j = \langle y_j^{c}, w_j \rangle$$

$k_i \geq 0, \ \forall i$ ⇒ $$z \in Z = \left\langle \sum_{j=1}^{n} k_j y_j^{c}, \sum_{j=1}^{n} |k_j| w_j \right\rangle$$



Calculate

$2 \times \langle 1, 0.75 \rangle + \langle 2, 1 \rangle.$

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach $x_1 \geq 0, \quad x_2 \geq 0$
  - Calculation of Possibilistic Linear Function value (Extension Principle)

$$f_0(x_1, x_2) = c_1 x_1 + c_2 x_2, \ \ C_1 = \langle 5, 1 \rangle, \ C_2 = \langle 7, 0.7 \rangle$$

$$\Rightarrow \ \ f_0(x_1, x_2) \in F_0(x_1, x_2) = \langle 5x_1 + 7x_2, \ x_1 + 0.7x_2 \rangle$$

In the same way, calculate: $f_i(x_1, x_2) = a_i x_1 + b_i x_2, \ a_i \in \tilde{A}_i, \ b_i \in \tilde{B}_i$

$$\tilde{A}_1 = \langle 2, 0.7 \rangle \ \ \tilde{B}_1 = \langle 3, 0.5 \rangle$$
$$\tilde{A}_2 = \langle 4, 1.5 \rangle \ \ \tilde{B}_2 = \langle 2, 0.3 \rangle$$
$$\tilde{A}_3 = \langle 1, 0.5 \rangle \ \ \tilde{B}_3 = \langle 3, 0.3 \rangle$$

$f_1(x_1, x_2) \in F_1(x_1, x_2)$?

$f_2(x_1, x_2) \in F_2(x_1, x_2)$?

$f_3(x_1, x_2) \in F_3(x_1, x_2)$?

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Inequality Indices based on Possibility Theory

To treat a possibilistic programming problem:

- The meaning of maximization of a fuzzy (possibilistic) function
- The meaning of the fact that a fuzzy (possibilistic) function value is not greater than 240.

Possibility and Certainty Degree of $\alpha \leq g$ (α: possibilistic variable)

$$Pos(\alpha \leq g) = \Pi_A((-\infty, g]) = \sup\{\mu_A(r) \mid r \leq g\}$$

$$Nes(\alpha \leq g) = N_A((-\infty, g]) = 1 - \sup\{\mu_A(r) \mid r > g\}$$

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Inequality Indices based on Possibility Theory

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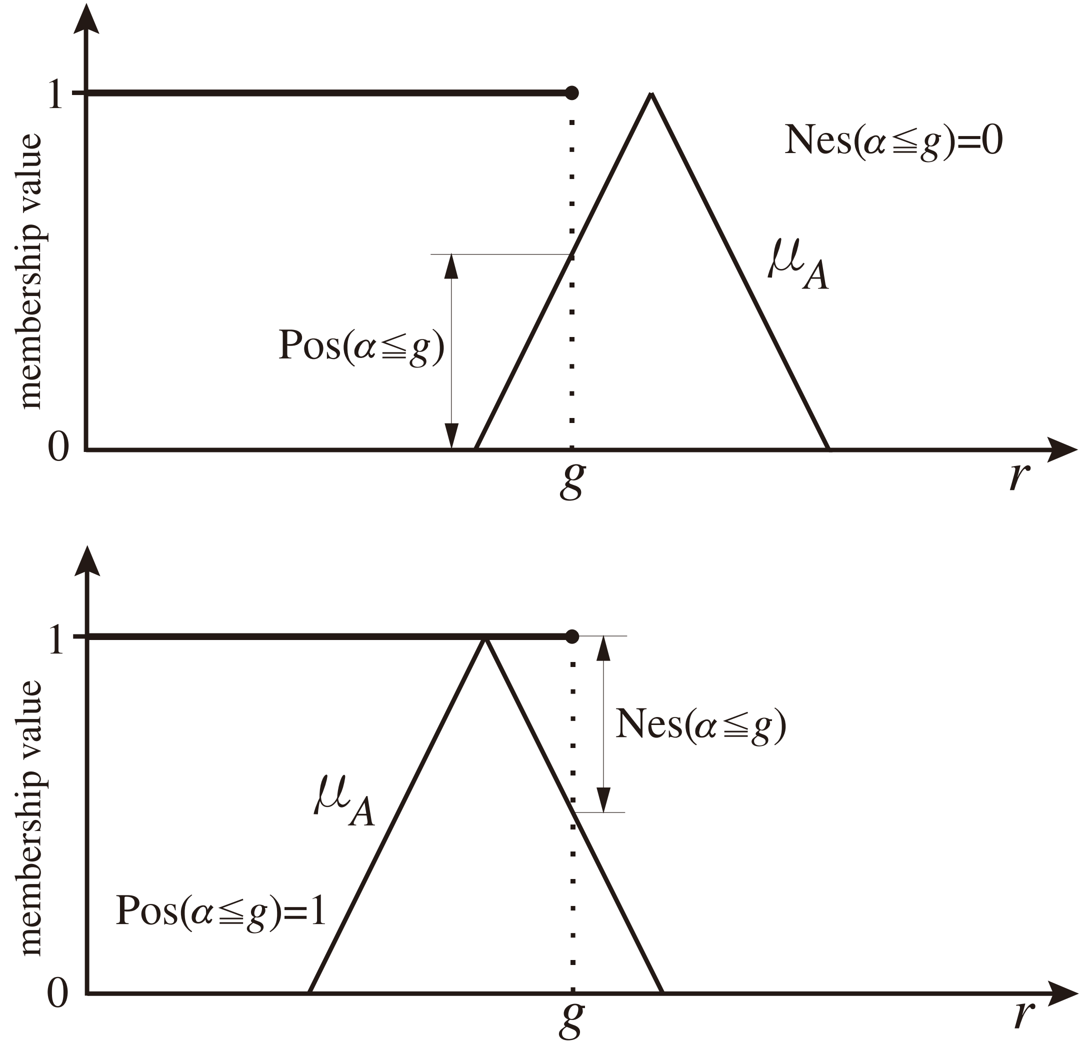


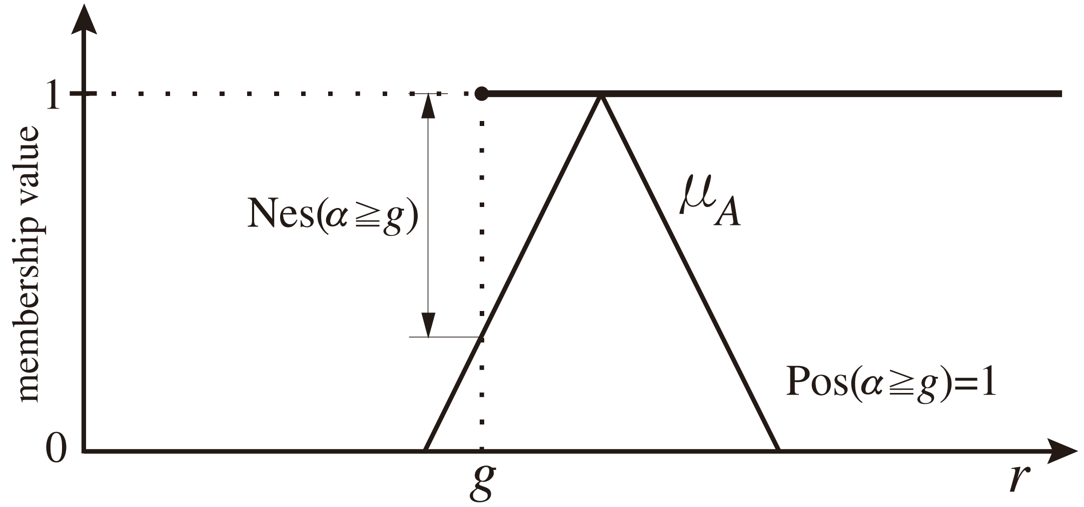
How to Use Fuzzy (Possibilistic) Programming ?

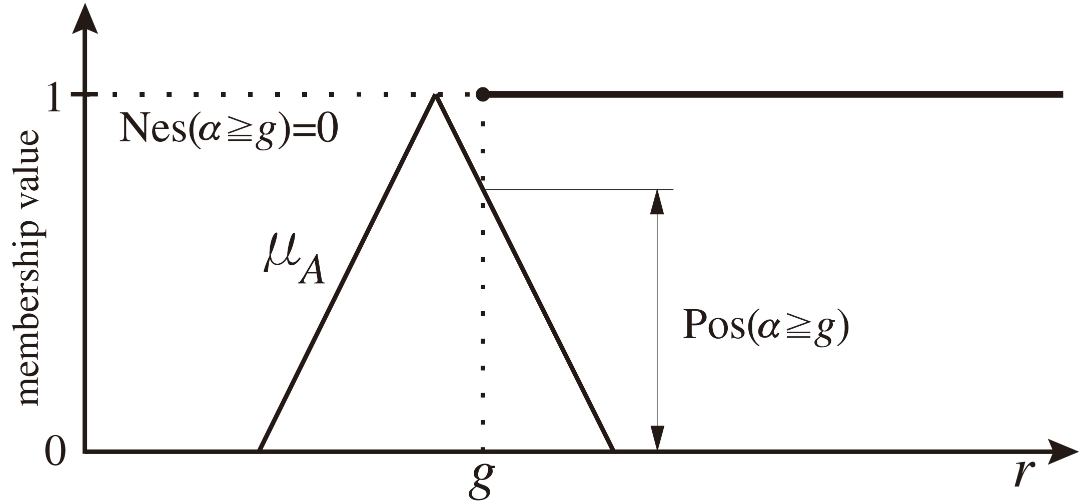
- Possibilistic (Fuzzy) Programming Approach
  - Inequality Indices based on Possibility Theory

Possibility and Certainty Degree of $\alpha \geq g$ (α: possibilistic variable)

$$Pos(\alpha \geq g) = \Pi_A([g, +\infty)) = \sup\{\mu_A(r) \mid r \geq g\}$$

$$Nes(\alpha \geq g) = N_A([g, +\infty)) = 1 - \sup\{\mu_A(r) \mid r < g\}$$





How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Treatment of Constraints

Assume that each working time cannot be extended for some reasons, such as the limited workshop space even if part-time workers are employed. In such a case, the constraints should be satisfied with high certainty (e.g., 0.8).

$$\begin{aligned}
&\mathrm{Nes}(a_1x_1 + b_1x_2 \leq 240) \geq 0.8,\\
&\mathrm{Nes}(a_2x_1 + b_2x_2 \leq 400) \geq 0.8,\\
&\mathrm{Nes}(a_3x_1 + b_3x_2 \leq 210) \geq 0.8,\\
&x_1 \geq 0,\ \ x_2 \geq 0.
\end{aligned}$$

$$\begin{array}{ll}
\tilde{A}_1 = \langle 2, 0.7\rangle & \tilde{B}_1 = \langle 3, 0.5\rangle\\
\tilde{A}_2 = \langle 4, 1.5\rangle & \tilde{B}_2 = \langle 2, 0.3\rangle\\
\tilde{A}_3 = \langle 1, 0.5\rangle & \tilde{B}_3 = \langle 3, 0.3\rangle
\end{array}$$

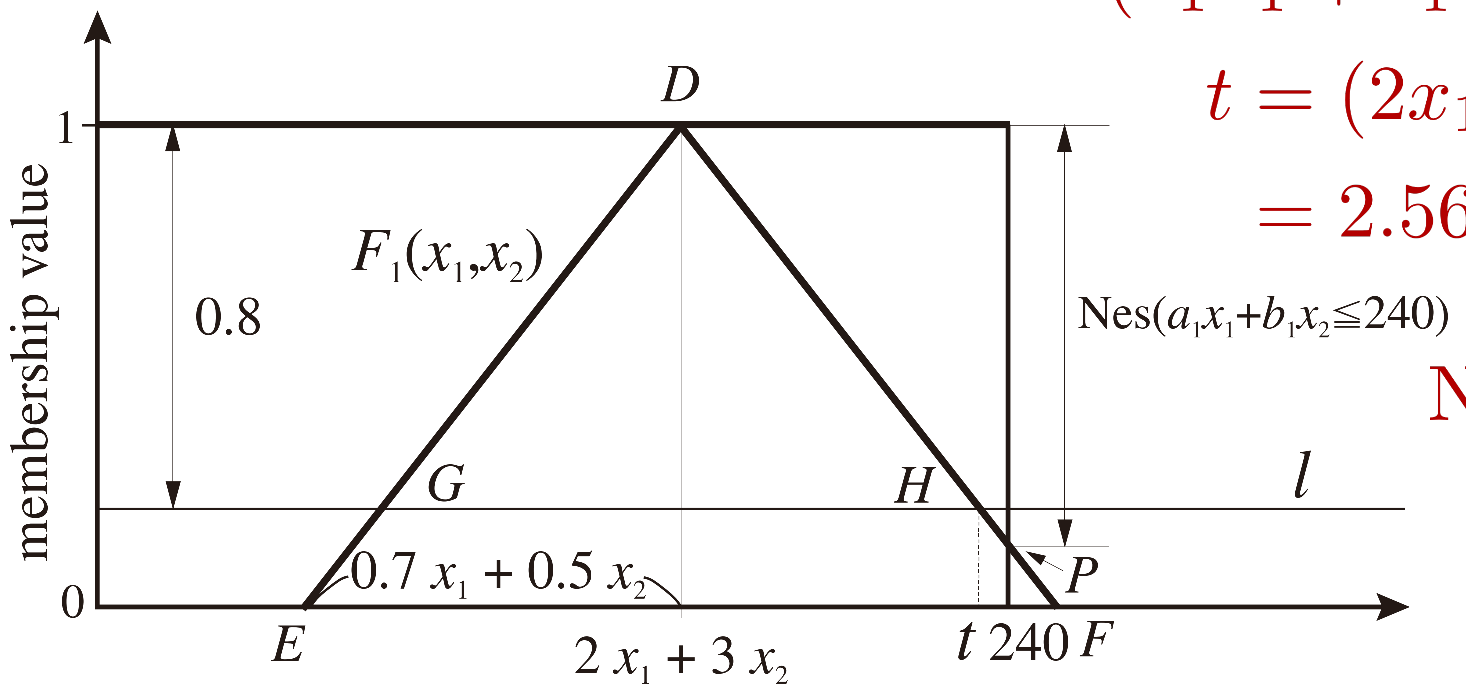
How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Treatment of Constraints

Analysis of $\mathrm{Nes}(a_1 x_1 + b_1 x_2 \leq 240) \geq 0.8$

$$\mathrm{Nes}(a_1 x_1 + b_1 x_2 \leq 240) \geq 0.8 \iff t \leq 240$$

$$t = (2x_1 + 3x_2) + 0.8(0.7x_1 + 0.5x_2)$$
$$= 2.56x_1 + 3.4x_2$$



$$\mathrm{Nes}(a_1 x_1 + b_1 x_2 \leq 240) \geq 0.8$$

⇕

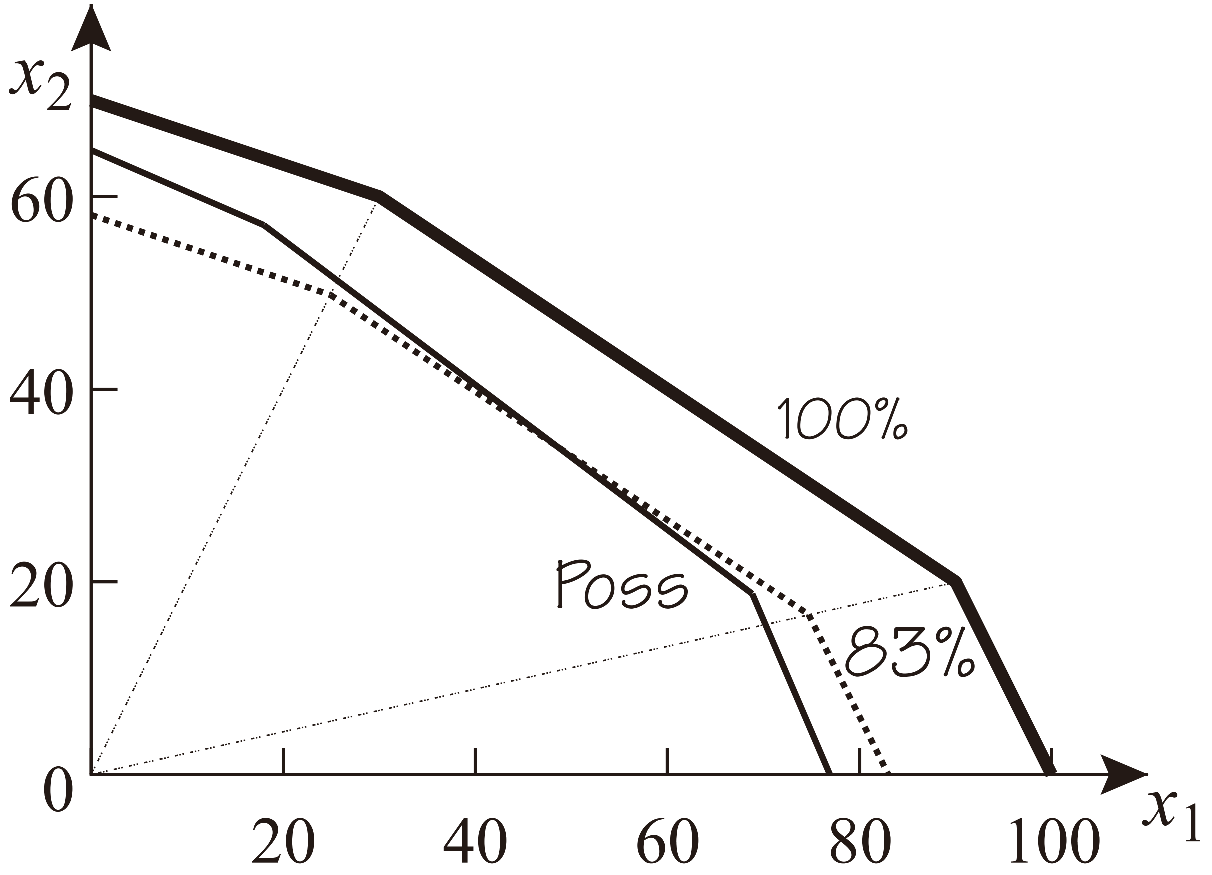
$$2.56x_1 + 3.4x_2 \leq 240$$

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Treatment of Constraints

$$\mathrm{Nes}(a_1x_1 + b_1x_2 \le 240) \ge 0.8,$$
$$\mathrm{Nes}(a_2x_1 + b_2x_2 \le 400) \ge 0.8,$$
$$\mathrm{Nes}(a_3x_1 + b_3x_2 \le 210) \ge 0.8,$$
$$x_1 \ge 0,\ \ x_2 \ge 0.$$

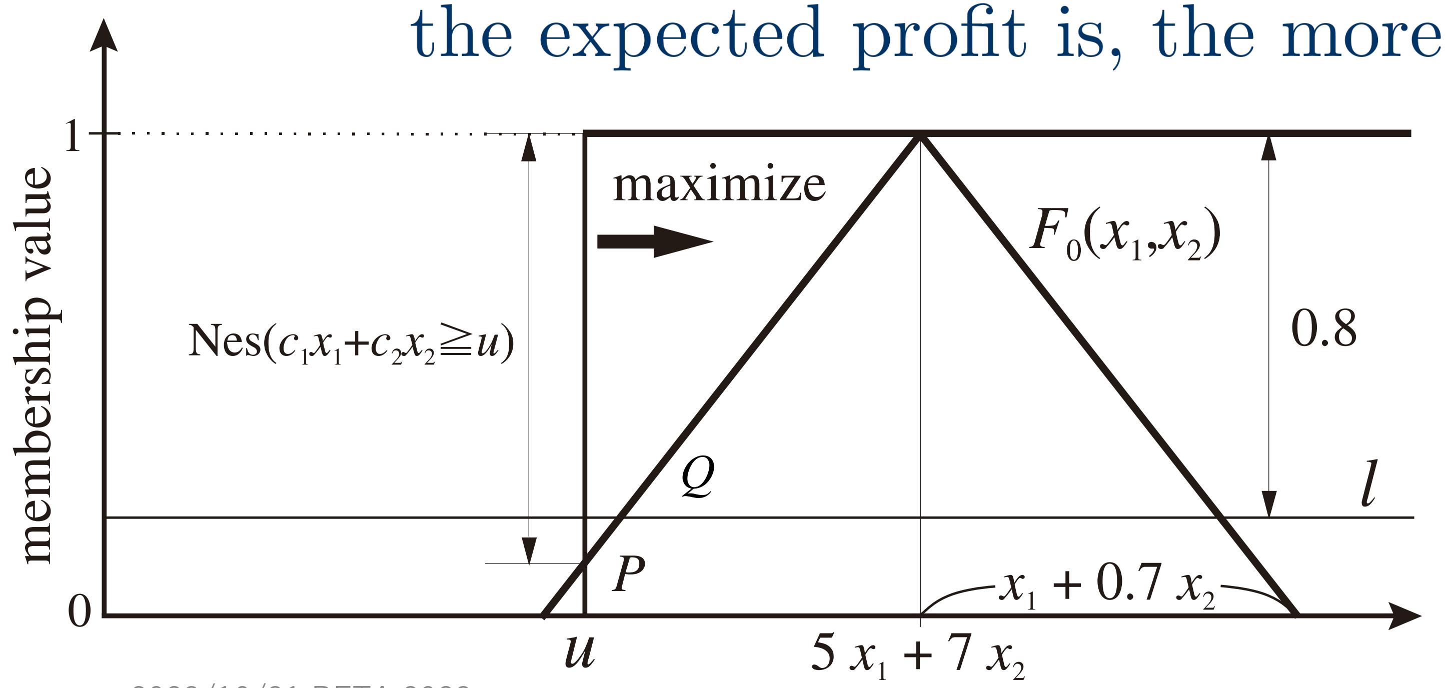
$$2.56x_1 + 3.4x_2 \le 240,$$
$$5.2x_1 + 2.24x_2 \le 400,$$
$$1.4x_1 + 3.24x_2 \le 210,$$
$$x_1 \ge 0,\ \ x_2 \ge 0.$$



How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Treatment of the Objective Function
      - Fractile Optimization Model (Value at Certainty Degree 0.8)

Assume that the decision maker has a great interest in the expected profit with high certainty (e.g., 0.8). Of course, the larger the expected profit is, the more preferable the solution is.



maximize u,
subject to
$\mathrm{Nes}(c_1x_1 + c_2x_2 \geq u) \geq 0.8.$

Maximize u under Q is located at the right side of P.

maximize $4.2x_1 + 6.44x_2$

How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Treatment of the Objective Function
      - Modality Optimization Model

Assume that the decision maker wants to maximize the certainty degree of the event that the profit is not smaller than 45,000 USD.

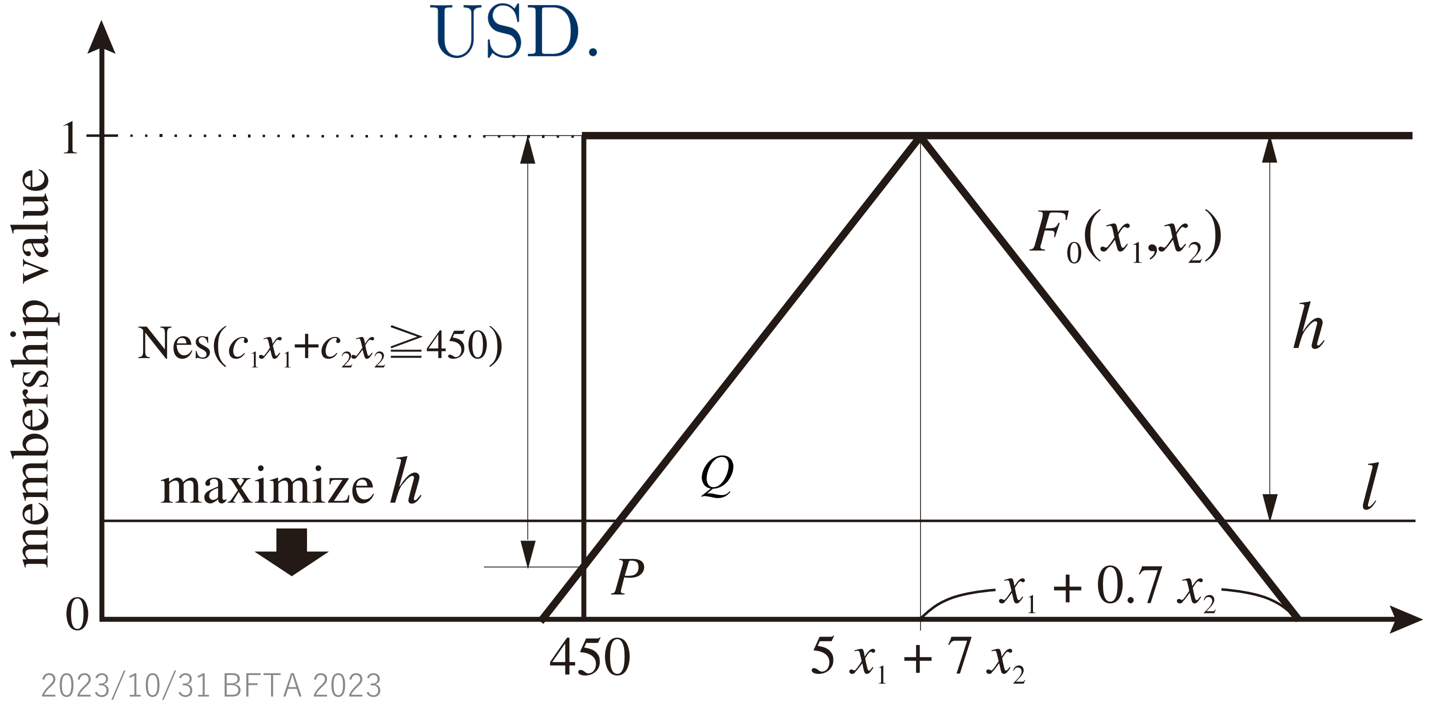
maximize

$$\mathrm{Nes}(c_1x_1 + c_2x_2 \geq 450).$$

Maximize h under Q is located at the right side of P.

$$\text{maximize} \quad \frac{5x_1 + 7x_2 - 450}{x_1 + 0.7x_2}$$

Because of the same reference function, we have the linear fractional objective function.



How to Use Fuzzy (Possibilistic) Programming ?

- Possibilistic (Fuzzy) Programming Approach
  - Formulation of the Possibilistic Linear Programming Problem
    - Fractile Optimization Model

$$
\begin{array}{ll}
\text{maximize} & 4.2x_1 + 6.44x_2, \\
\text{subject to} & 2.56x_1 + 3.4x_2 \le 240, \\
 & 5.2x_1 + 2.24x_2 \le 400, \\
 & 1.4x_1 + 3.24x_2 \le 210, \\
 & x_1 \ge 0,\ x_2 \ge 0.
\end{array}
$$

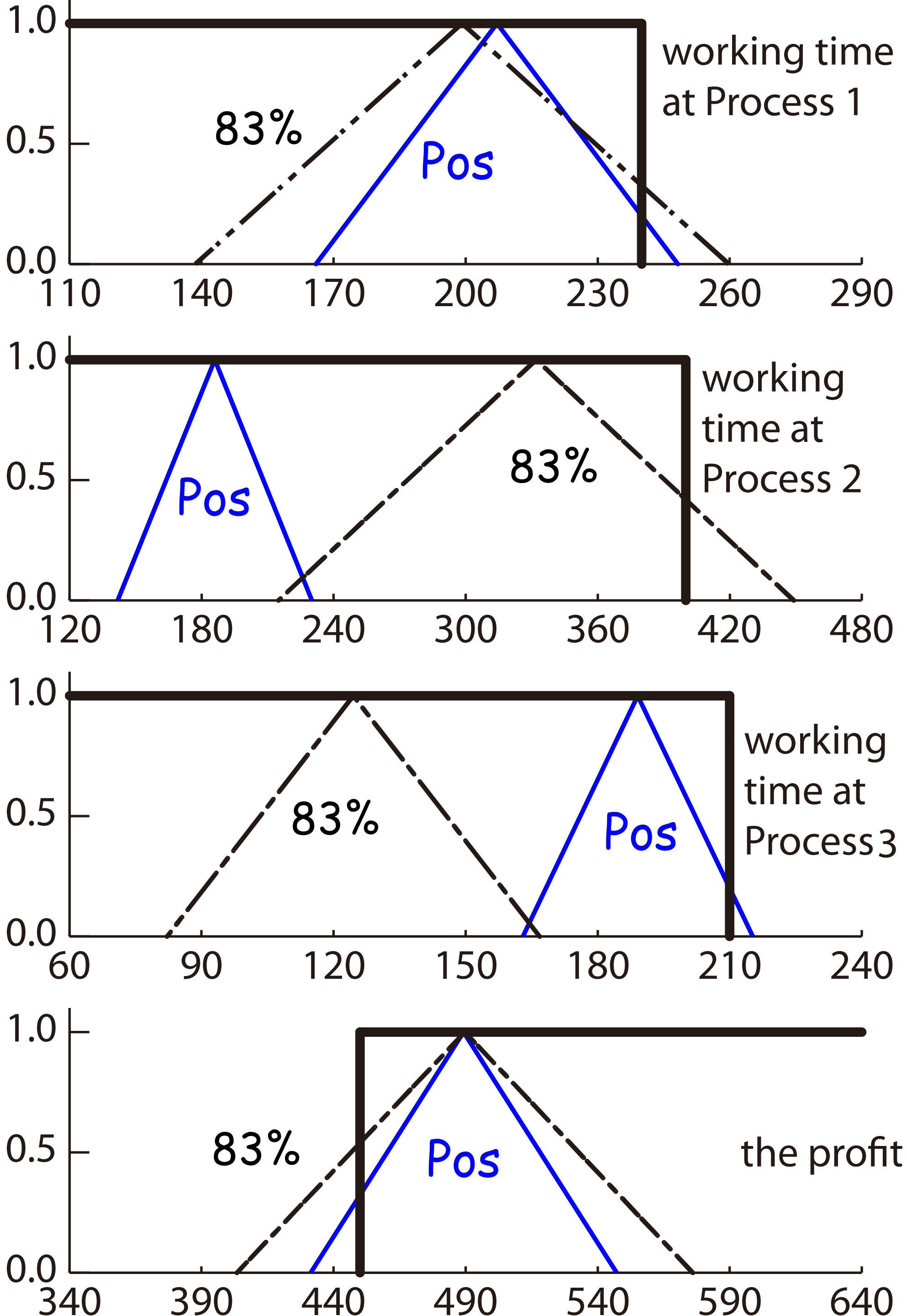
$.99, 57.04)^{\mathrm{T}}$

$$
\begin{array}{ll}
\text{maximize} & 5y_1 + 7y_2 - 450t \\
\text{subject to} & 2.56y_1 + 3.4y_2 \le 240t, \\
 & 5.2y_1 + 2.24y_2 \le 400t, \\
 & 1.4y_1 + 3.24y_2 \le 210t, \\
 & y_1 + 0.7y_2 = 1, \\
 & y_1 \ge 0,\ y_2 \ge 0,\ t \ge 0.
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & \dfrac{5x_1 + 7x_2 - 450}{x_1 + 0.7x_2}, \\
\text{subject to} & 2.56x_1 + 3.4x_2 \le 240, \\
 & 5.2x_1 + 2.24x_2 \le 400, \\
 & 1.4x_1 + 3.24x_2 \le 210, \\
 & x_1 \ge 0,\ x_2 \ge 0.
\end{array}
$$

Solution: $(x_1, x_2)^{\mathrm{T}} \approx (17.99, 57.04)^{\mathrm{T}}$

How to Use Fuzzy (Possibilistic) Programming ?



- Comparison of Solutions
- The solution to 83%-Problem: The certainty degree of the satisfaction of constraints on working time at Process 1 and Process 2 is not high enough.
- Thus, we may regard the solution to 83%-Problem as an ill-matched solution to the decision maker's intention.

How to Use Fuzzy (Possibilistic) Programming ?

$$\begin{aligned}
\text{maximize} \quad & h, \\
\text{subject to} \quad & (5-h)x_1 + (7-0.7h)x_2 \geq 450, \\
& (6-h)x_1 + (7.7-0.7h)x_2 \geq 530, \\
& 2.56x_1 + 3.4x_2 \leq 240, \\
& 5.2x_1 + 2.24x_2 \leq 400, \\
& 1.4x_1 + 3.24x_2 \leq 210, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned}$$

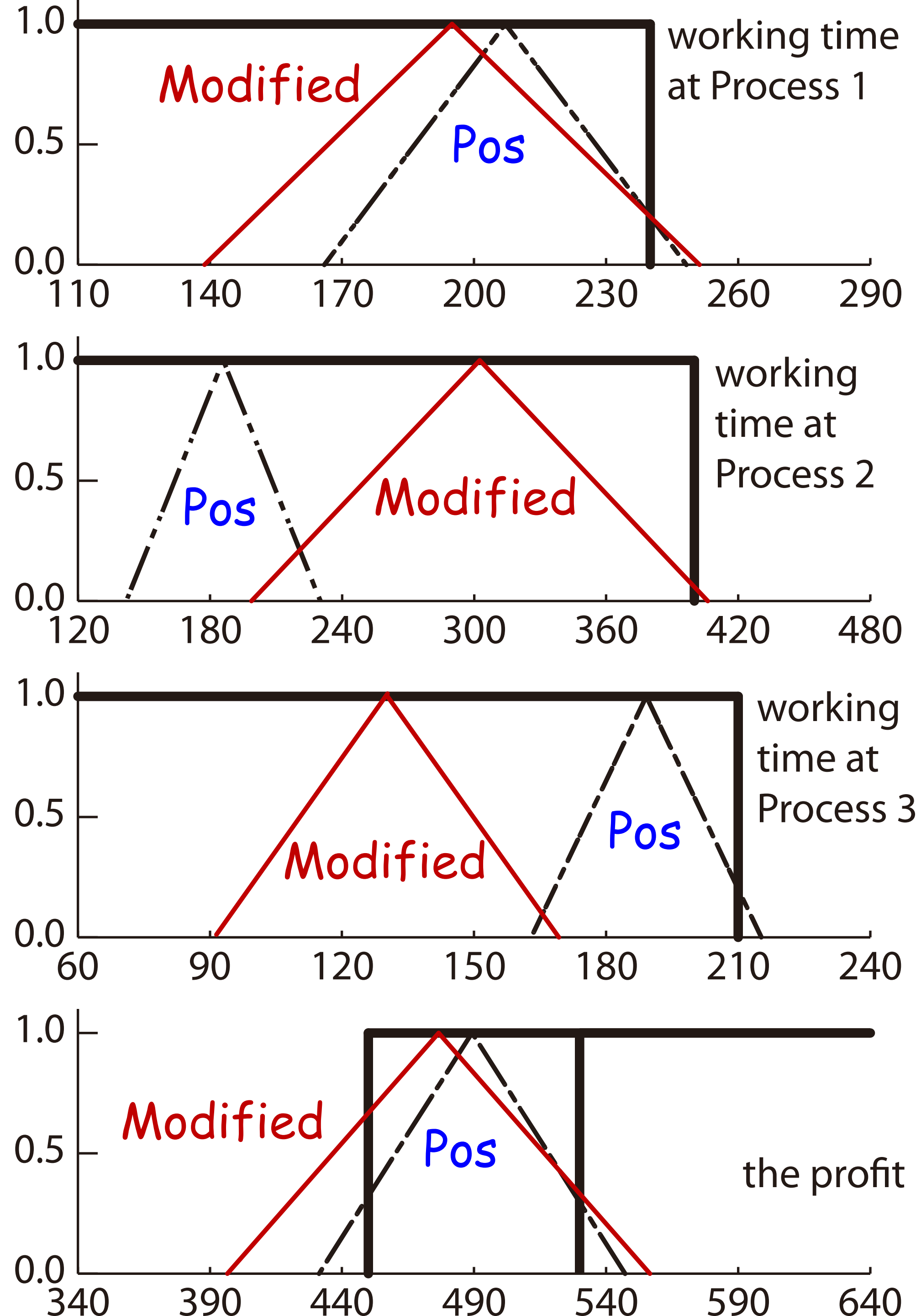
$$\begin{aligned}
\text{maximize} \quad & h, \\
\text{subject to} \quad & 5y_1 + 7y_2 - 450t \geq h, \\
& 6y_1 + 7.7y_2 - 530t \geq h, \\
& 2.56y_1 + 3.4y_2 \leq 240t, \\
& 5.2y_1 + 2.24y_2 \leq 400t, \\
& 1.4y_1 + 3.24y_2 \leq 210t, \\
& y_1 + 0.7y_2 = 1, \\
& y_1 \geq 0, \ y_2 \geq 0, \ t \geq 0.
\end{aligned}$$

- Comparison of Solutions

Assume that the decision maker is not satisfied with the solution of Poss. If he/she requires that the possibility degree of the event that the profit is not smaller than 53,000 USD is as high as the necessity degree of the event that the profit is not smaller than 45,000 USD, we can Reformulate the objective function as

$$\begin{aligned}
\text{maximize} \quad \min(&\text{Nes}(c_1x_1 + c_2x_2 \geq 450), \\
&\text{Pos}(c_1x_1 + c_2x_2 \geq 530)).
\end{aligned}$$

Solution: $(x_1, x_2)^{\mathrm{T}} \approx (64.68, 21.89)^{\mathrm{T}}$



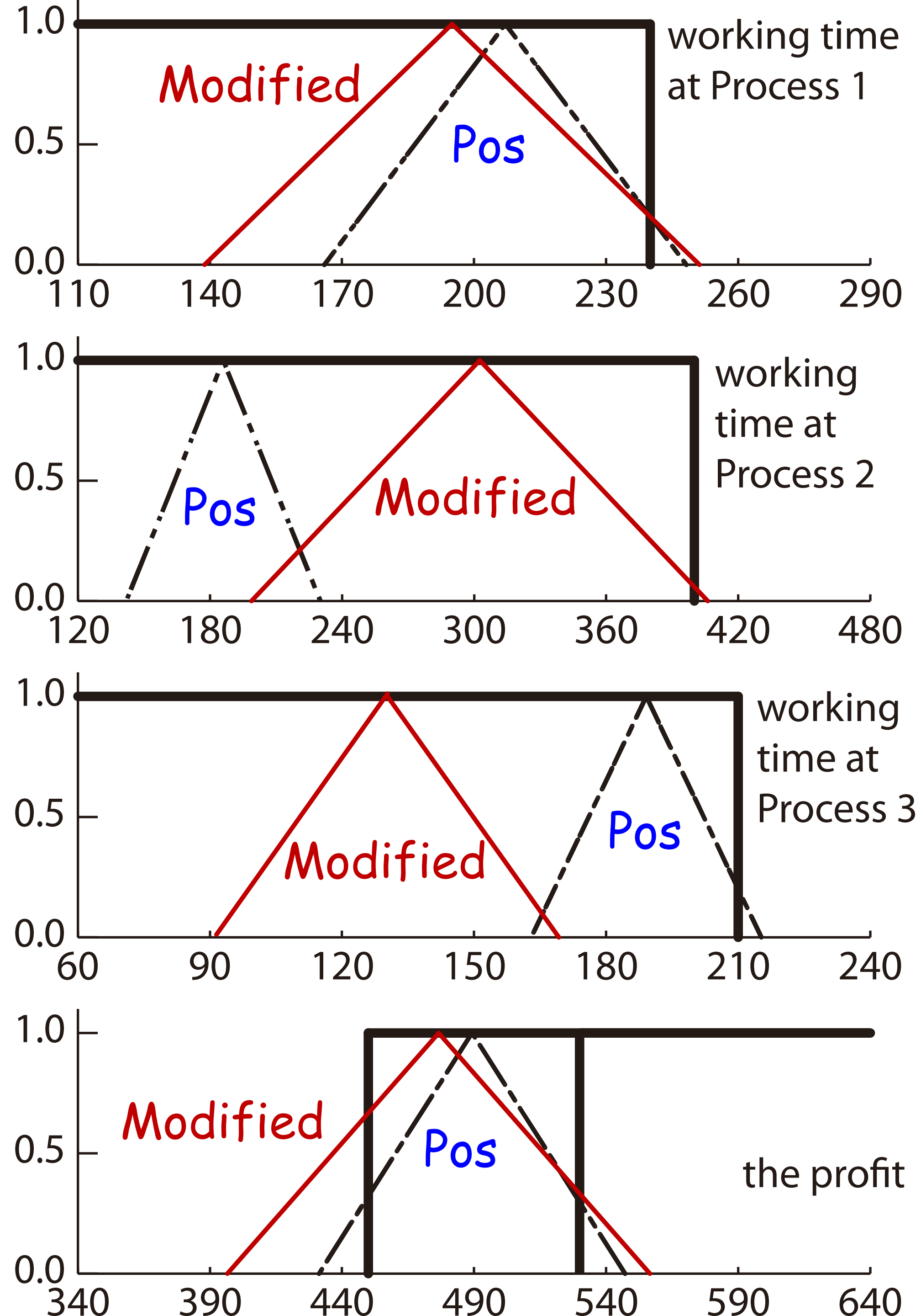
How to Use Fuzzy (Possibilistic) Programming ?

- Comparison of Solutions

Assume that the decision maker is not satisfied with the solution of Poss. If he/she requires that the possibility degree of the event that the profit is not smaller than 53,000 USD is as high as the necessity degree of the event that the profit is not smaller than 45,000 USD, we can Reformulate the objective function as

$$\begin{aligned}\text{maximize}\quad &\min(\text{Nes}(c_1x_1+c_2x_2\geq 450),\\ &\text{Pos}(c_1x_1+c_2x_2\geq 530)).\end{aligned}$$

Solution: $(x_1, x_2)^{\mathrm{T}} \approx (64.68, 21.89)^{\mathrm{T}}$



How to Use Fuzzy (Possibilistic) Programming ?

- Comparison of Solutions

As shown in Figure, compared to Pos, Modified makes the possibility degree of the event that the profit is not smaller than 53,000 USD a little bit higher but it makes the certainty degree of the event that the profit is not smaller than 45,000 USD lower. The decision maker may know that he cannot offer a higher requirement than the solution to Problems Pos and Modified.

How to Use Fuzzy ... ming ?

Brief Review

- Possibilistic (fuzzy) Programming Approach
- Formulations & Reduced Problems
- Various Solutions reflecting DM's intension
- **The reduced problems are simpler than those of stochastic programming approach.**

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Pos

$(x_1, x_2)^{\mathrm{T}} \approx ($ ... $0)),$

Modified

$(x_1, x_2)^{\mathrm{T}} \approx ($ ... $0) \geq 0.5.$

Solution: $(x_1, x_2)^{\mathrm{T}} \approx (38.28, 41.76)^{\mathrm{T}}$

Intermediate

Possibility Theory
(Possibility and Necessity Measures)

- Extension of Inequality Relation for Fuzzy Numbers based on possibility and necessity measures (Dubois & Prade, 1983)
  - Four extended inequality relations
  - Inequality relation → fuzzy inequality relation (preference relation)
    (Inuiguchi et al., 1991)
- Possible and Necessary Optimality
  - Extension of optimality by possibility and necessity measures
    (Inuiguchi & Sakawa, 1994)

  Optimality → Soft-optimality = Minimax regret approach
  (Inuiguchi & Sakawa, 1995 & 1998)
  - Optimality → Efficiency (Pareto Optimality)
    (Inuiguchi & Sakawa, 1996)
- Etc.

Possibly and necessarily optimal solutions

Example 1:

maximize $\gamma^\top \boldsymbol{x}$, subject to $A\boldsymbol{x} \le \boldsymbol{b}$; $\gamma \in \Gamma$

where

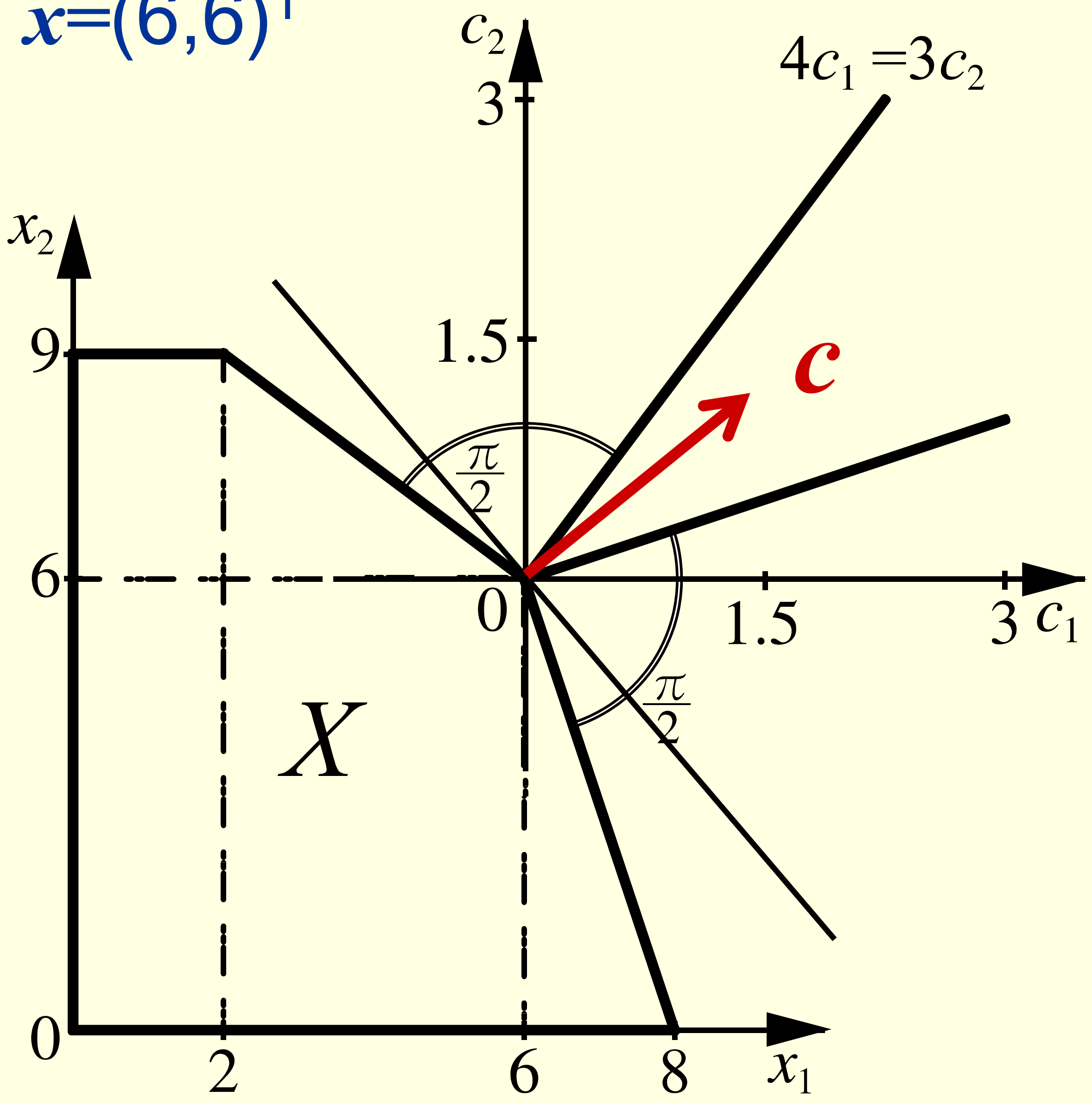
$$A = \begin{pmatrix} 3 & 3 & 0 & -1 & 0 \\ 4 & 1 & 1 & 0 & -1 \end{pmatrix}^\top,$$

$$\boldsymbol{b} = (42, 24, 9, 0, 0)^\top,$$

$$\Gamma = \{(c_1, c_2)^\top : 3.5 \le 2c_1 + c_2 \le 5.5,\ 3.4 \le c_1 + 2c_2 \le 6,\ 1 \le c_1 - c_2 \le 1.3,\ 1 \le c_1 \le 2,\ 0.8 \le c_2 \le 2.2\}.$$

Optimal Solution in LP Problem with *c*

Example 1: $\boldsymbol{x}=(6,6)^T$

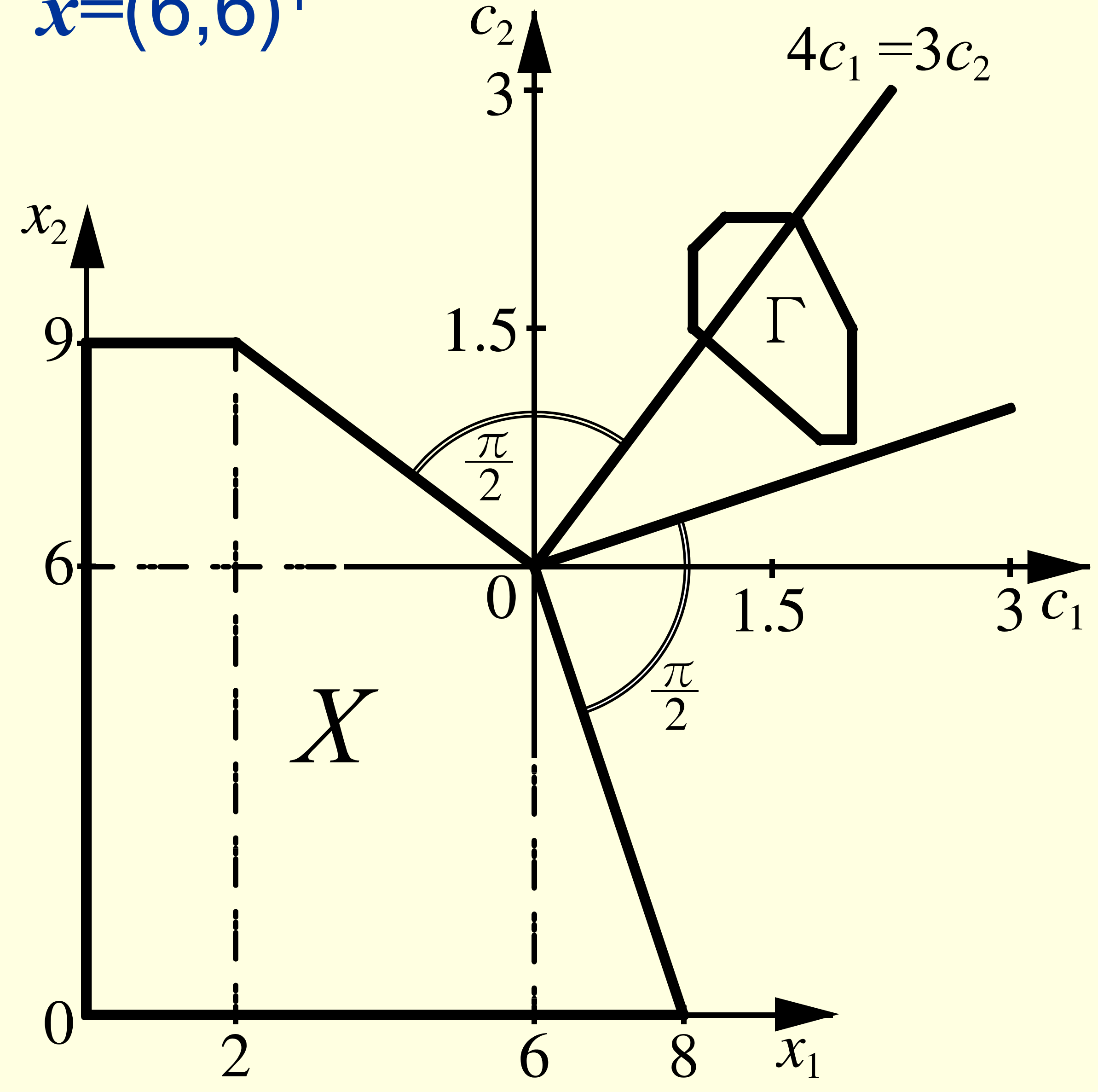


Possibly and necessarily optimal solutions

Example 1: $\boldsymbol{x}=(6,6)^{\mathrm{T}}$

Possibly optimal solution

Necessarily optimal solution



Possibly and necessarily optimal solutions

Example 2:

maximize $\gamma^{\top}\boldsymbol{x}$, subject to $A\boldsymbol{x} \le \boldsymbol{b}$; $\gamma \in \Gamma$

where

$$A = \begin{pmatrix} 3 & 3 & 0 & -1 & 0 \\ 4 & 1 & 1 & 0 & -1 \end{pmatrix}^{\top},$$

$$\boldsymbol{b} = (42, 24, 9, 0, 0)^{\top},$$

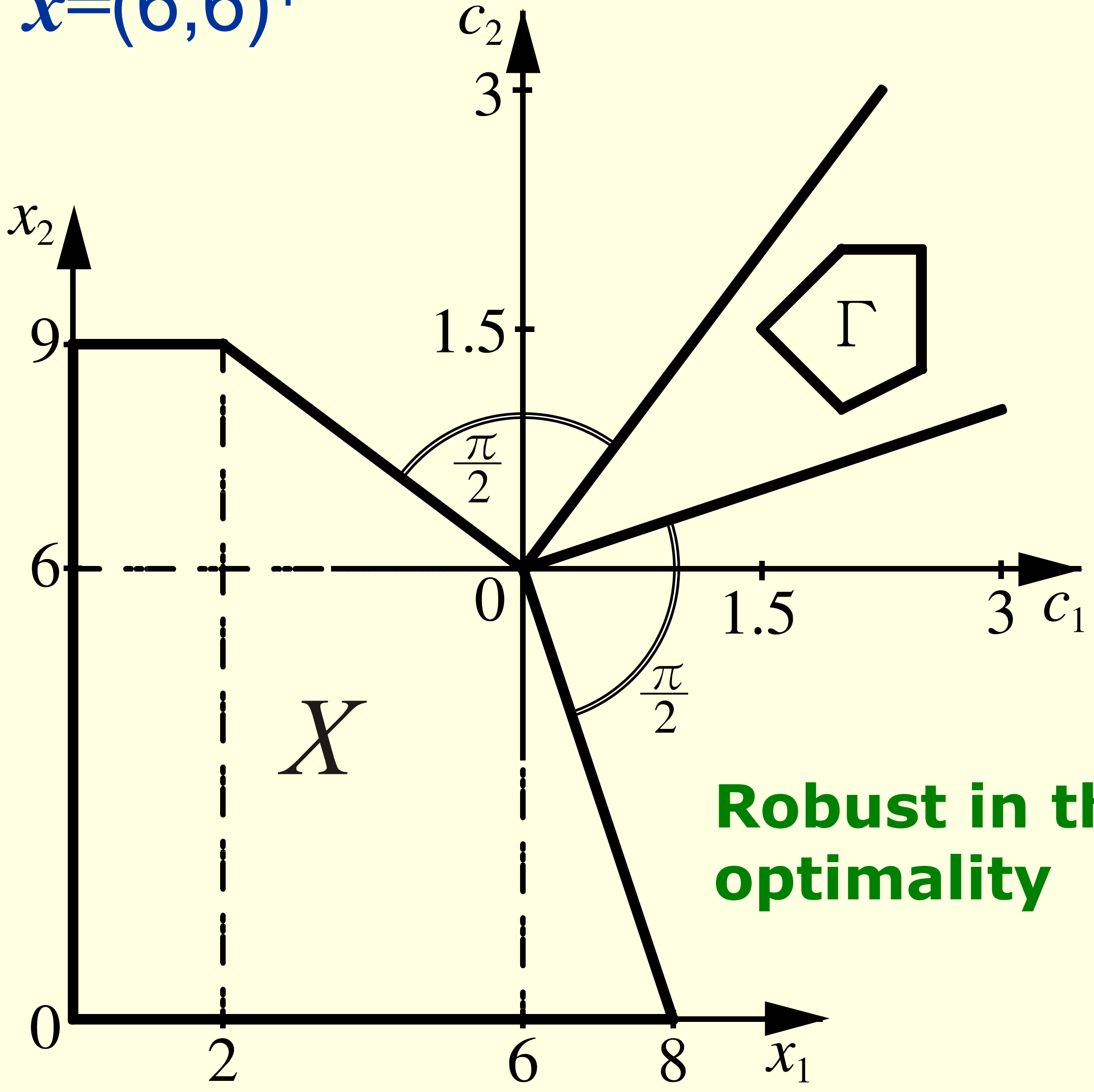
$$\Gamma = \{(c_1, c_2)^{\top} : c_1 + c_2 \ge 3, \\ c_1 \ge c_2,\ c_1 \le 2c_2,\ c_1 \le 2.5,\ c_2 \le 2\}.$$

Possibly and necessarily optimal solutions

Example 2: $\boldsymbol{x}=(6,6)^{\mathrm{T}}$

Possibly optimal solution

Necessarily optimal solution

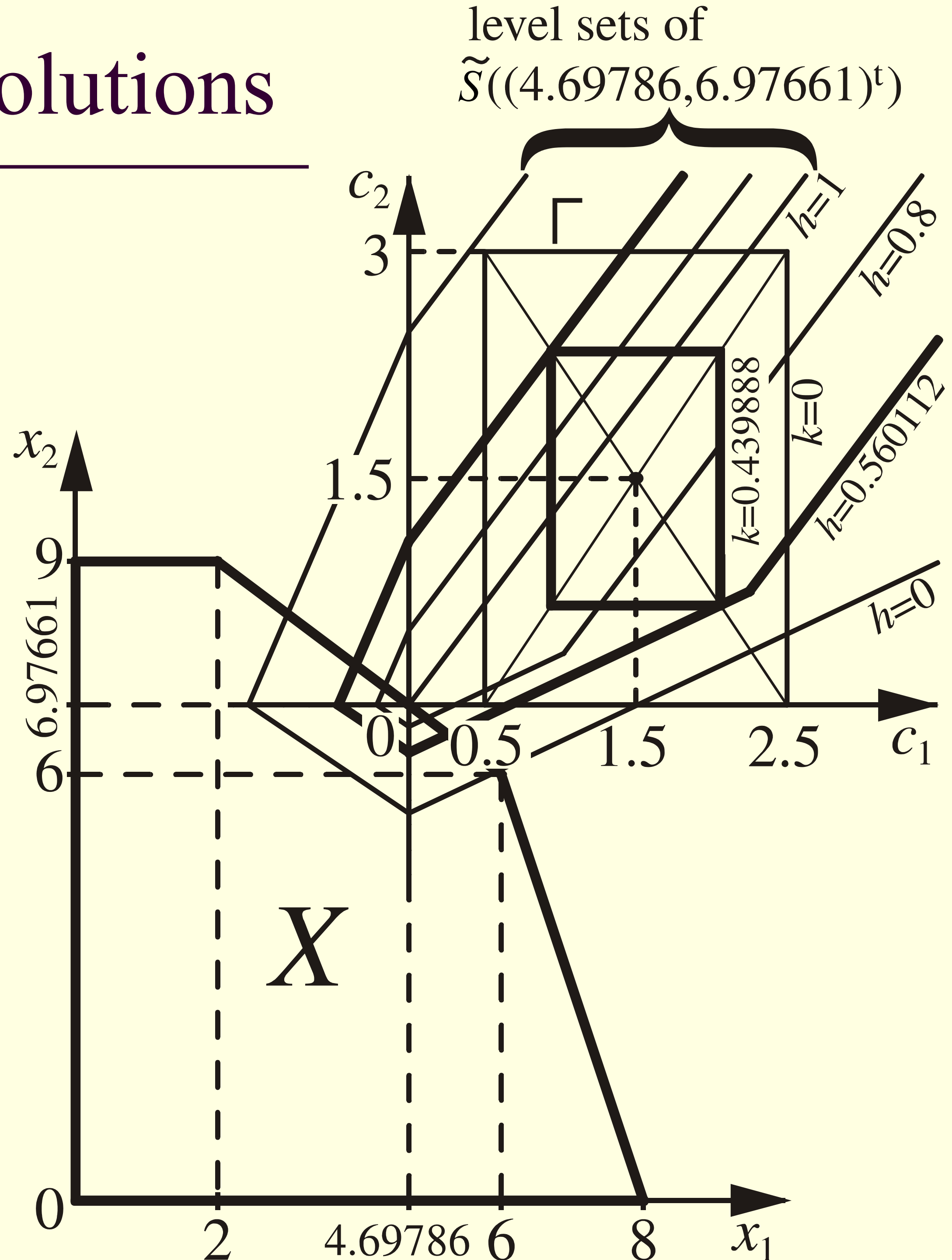


Robust in the optimality

Best necessarily soft optimal solutions

Example:

Necessarily soft optimal solution to degree 0.560112



Necessity Measure

- Definition of Necessity Measure:

$$N_A(B) = \inf_{r \in U} I(\mu_A(r), \mu_B(r)),$$

μ_A : membership function of A

μ_B : membership function of B

  - I: $[0,1]\times[0,1]\to[0,1]$: an implication function,

(I0) I is upper semi-continuous, (semi-continuity)

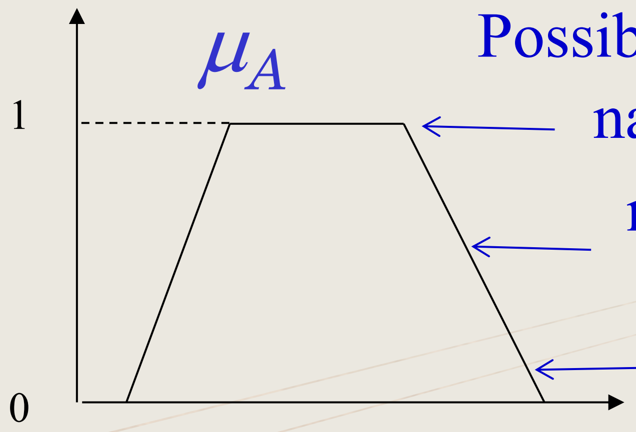
(I1) $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$, (boundary condition)

(I2) $I(a,b) \le I(c,d)$ if $0 \le c \le a \le 1$ and $0 \le b \le d \le 1$. (monotonicity)

Necessity measure shows a degree of inclusion $A \subseteq B$.

Necessity Measures

Fuzzy set



Possible range:

narrowest estimated range

medium-sized estimated range

widest estimated range

For $h \in [0,1]$,

h-level set

$[A]_h = \{r \mid \mu_A(r) \geq h\}$

strong h-level set

$(A)_h = \{r \mid \mu_A(r) > h\}$

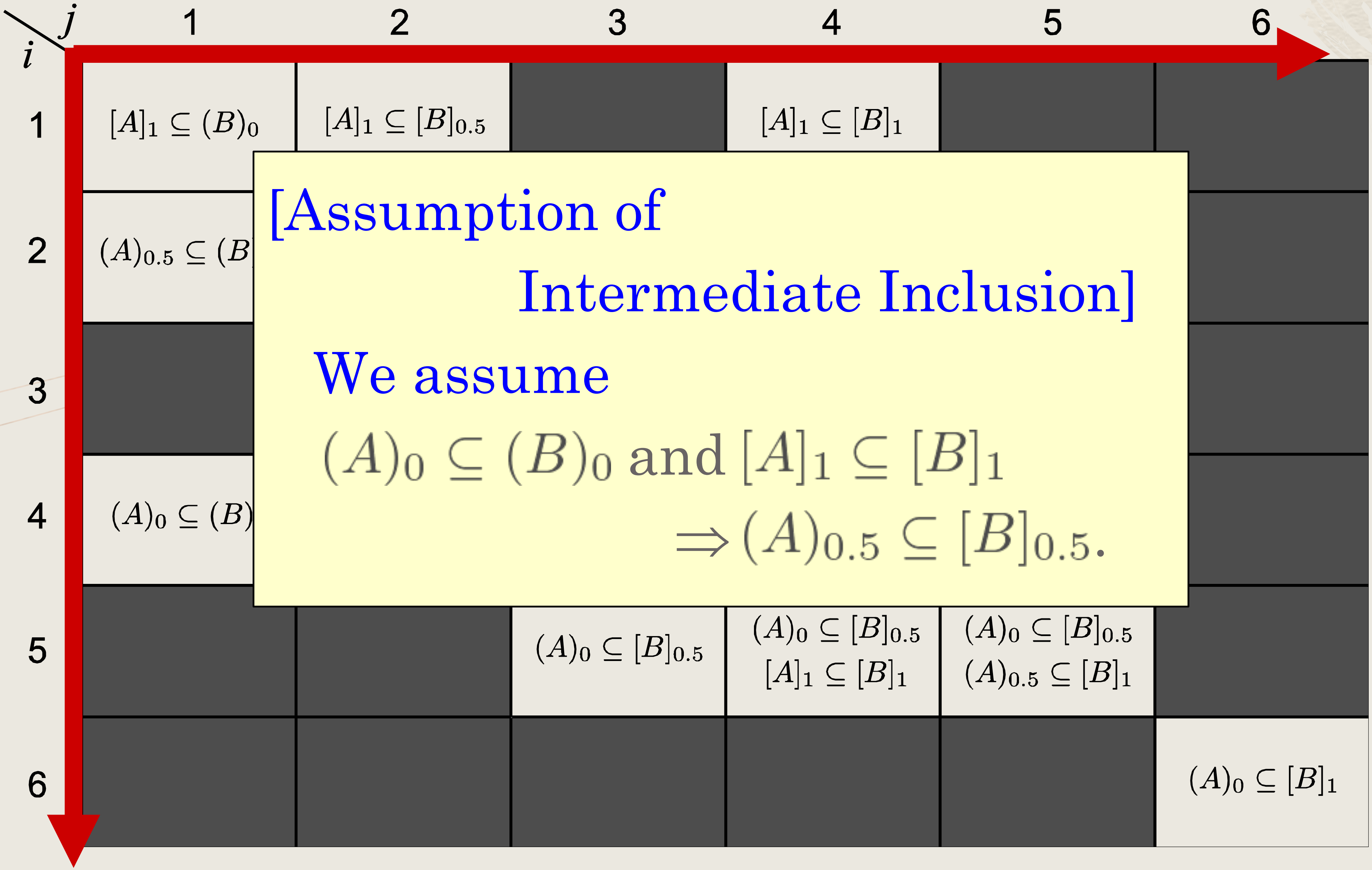
$[A]_1$ 1 narrowest estimated range 1 $[B]_1$

$(A)_{0.5}$ 0.5 Medium-sized estimated range 0.5 $[B]_{0.5}$

$(A)_0$ +0 Widest estimated range +0 $(B)_0$

$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{x}$ B_i

We give an enhancing **sequence of conditions about** $\boldsymbol{A}_i^T\boldsymbol{x} \subseteq \boldsymbol{B}_i$ using those 6 ranges (3 ranges for each) to express DM's requirement on robust condition.



j
i
1
2
3
4
5
6
1
2
3
4
5
6
$[A]_1 \subseteq (B)_0$
$[A]_1 \subseteq [B]_{0.5}$
$[A]_1 \subseteq [B]_1$
$(A)_{0.5} \subseteq (B$
$(A)_0 \subseteq (B)$
$(A)_0 \subseteq [B]_{0.5}$
$(A)_0 \subseteq [B]_{0.5}$ $[A]_1 \subseteq [B]_1$
$(A)_0 \subseteq [B]_{0.5}$ $(A)_{0.5} \subseteq [B]_1$
$(A)_0 \subseteq [B]_1$
[Assumption of Intermediate Inclusion]
We assume
$(A)_0 \subseteq (B)_0$ and $[A]_1 \subseteq [B]_1$
$\Rightarrow (A)_{0.5} \subseteq [B]_{0.5}.$

$Inc(i, j)$

i \ j	1	2	3	4	5	6
1	$[A]_1 \subseteq (B)_0$	$[A]_1 \subseteq [B]_{0.5}$		$[A]_1 \subseteq [B]_1$		
2	$(A)_{0.5} \subseteq (B)_0$	$(A)_{0.5} \subseteq (B)_0$ $[A]_1 \subseteq [B]_{0.5}$		$(A)_{0.5} \subseteq (B)_0$ $[A]_1 \subseteq [B]_1$		
3			$(A)_{0.5} \subseteq [B]_{0.5}$	$(A)_{0.5} \subseteq [B]_{0.5}$ $[A]_1 \subseteq [B]_1$	$(A)_{0.5} \subseteq [B]_1$	
4	$(A)_0 \subseteq (B)_0$	$(A)_0 \subseteq (B)_0$ $[A]_1 \subseteq [B]_{0.5}$	$(A)_0 \subseteq (B)_0$ $(A)_{0.5} \subseteq [B]_{0.5}$	$(A)_0 \subseteq (B)_0$ $(A)_{0.5} \subseteq [B]_{0.5}$ $[A]_1 \subseteq [B]_1$	$(A)_0 \subseteq (B)_0$ $(A)_{0.5} \subseteq [B]_1$	
5			$(A)_0 \subseteq [B]_{0.5}$	$(A)_0 \subseteq [B]_{0.5}$ $[A]_1 \subseteq [B]_1$	$(A)_0 \subseteq [B]_{0.5}$ $(A)_{0.5} \subseteq [B]_1$	
6						$)_0 \subseteq [B]_1$

DM expresses his/her preference on the robustness from most necessary requirement to the favorable requirement by a sequence of $Inc(i_1,j_1)$, ..., $Inc(i_p,j_p)$.

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Contents

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- Introduction to Possibility Theory
  - Possibility and necessity measures
  - Correspondence to Plausibility and Belief functions
- Application to Decision Making: Decision principles
  - Possibility measure maximization
  - Necessity measure maximization
  - Relative possibility measure maximization
- Possibilistic (Fuzzy) Linear Programming
  - (Much simpler than stochastic linear programming)