Measuring inconsistency in evidence theory

Anne-Laure Jousselme

CS Group - France

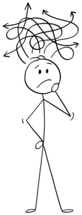
6th School on Belief Functions and Their Applications

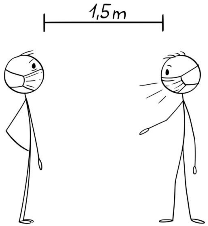
Japan Advanced Institute of Science and Technology (JAIST)
Ishikawa, Japan

Overview

Inconsistency measures (*Informational measures*)

- Unary (internal) $\delta(m)$
  - Entropy-like
  - Consistency-based
- Binary (external) $\delta(m_1, m_2)$
  - Conflict
  - Distance





Purpose

- Survey of internal and external measures of inconsistency
- . . . and generalizations

Outline

Outline

Outline

Notations

- $\mathcal{X}$ is the frame of discernment of cardinality n: $\mathcal{X} = \{x_1, \ldots, x_n\}$
- $\mathcal{P}(\mathcal{X})$ is its power set of cardinality 2^n: $\mathcal{P}(\mathcal{X}) = \{\emptyset, x_1, \ldots, x_n, (x_1, x_2), \ldots, \mathcal{X}\}$
- x is an element of $\mathcal{X}$: $x \in \mathcal{X}$
- A is a subset of $\mathcal{X}$, element of $\mathcal{P}(\mathcal{X})$: $A \subseteq \mathcal{X}$, $A \in \mathcal{P}(\mathcal{X})$
- $|A|$ is the cardinality of A
- $\overline{A}$ is the complement of A relatively to $\mathcal{X}$: $\overline{A} = \mathcal{X} \backslash A$
- $A \cap B$ denotes the intersection of A and B
- $A \cup B$ denotes the union of A and B
- m is a mass function; $\sum_{A \subseteq \mathcal{X}} m(A) = 1$
- $\mathcal{F} = \{A \subseteq \mathcal{X}; m(A) \neq 0\}$ is the set of focal sets of m
- $|\mathcal{F}|$ is the number of focal sets of m
- Bel is a belief function, Pl is the plausibility function

Special mass functions

- m is normalised if $m(\emptyset) = 0$
- m is Bayesian if all focal sets are singletons
- m is logical (or categorical) if $m(A) = 1$ for some $A \subseteq \mathcal{X}$. It is equivalent to A and denoted m_A:
  - $m_{\mathcal{X}}$ represents total ignorance (and is called vacuous)

$$m(\mathcal{X}) = 1$$

  - $m_\emptyset$ represents total inconsistency

$$m(\emptyset) = 1$$

CS GROUP

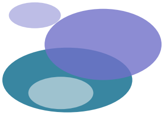
Outline

Set and probability dimensions

Belief functions extend both:

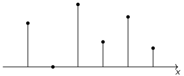
classical sets

$$m(A) = 1 \text{ for some } A \subseteq \mathcal{X}$$



probabilities

$$\sum_{x \in \mathcal{X}} m(\{x\}) = 1$$



Set and probability dimensions

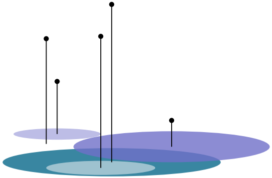
Belief functions extend both:

classical sets

$$m(A) = 1 \text{ for some } A \subseteq \mathcal{X}$$

probabilities

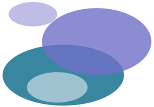
$$\sum_{x \in \mathcal{X}} m(\{x\}) = 1$$



$$m(\emptyset) \neq 0$$

Unary measures

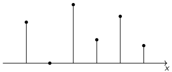
Hartley entropy



$$H(A) = \log |A|$$

- $H(\{x\}) = 0$

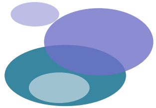
Shannon entropy



$$Sh(p) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$

- 0 if $p(\{x\}) = 1$
- Maximum for uniform distribution

Some basic measures



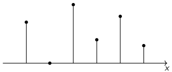
Unary measures

Hartley entropy

$$H(A) = \log |A|$$

- $H(\{x\}) = 0$

Shannon entropy



$$Sh(p) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$

- 0 if $p(\{x\}) = 1$
- Maximum for uniform distribution

Binary measures

Distance between sets

$$d(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

Euclidean distance

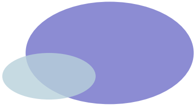
$$d(p_1, p_2) = \left(\sum_{x \in \mathcal{X}} (p_1(x) - p_2(x))^2 \right)^{\frac{1}{2}}$$

Kullback-Liebler divergence

$$KL(p_1 || p_2) = -\sum_{x \in \mathcal{X}} p_1(x) \log \frac{p_1(x)}{p_2(x)}$$

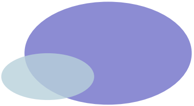
Outline

Two sets are consistent if their intersection is not empty



Consistency indexes

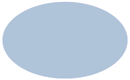
Two sets are consistent if their intersection is not empty

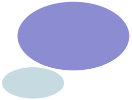


Consistency index

A binary consistency index between two sets is such that:

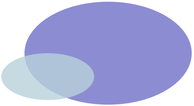
$$\phi(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ if } A \cap B = \emptyset \end{cases}$$





Consistency indexes

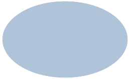
Two sets are consistent if their intersection is not empty

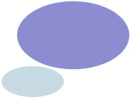


Consistency index

A binary consistency index between two sets is such that:

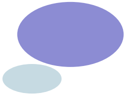
$$\phi(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ if } A \cap B = \emptyset \end{cases}$$



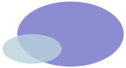


$\phi(A,B)$
$\phi_p(A,B) = \begin{cases} 1 \text{ if } A \cap B \neq \emptyset \\ 0 \text{ else} \end{cases}$
$\phi_b(A,B) = \begin{cases} 1 \text{ if } B \subseteq A \\ 0 \text{ else} \end{cases}$
$\phi_q(A,B) = \begin{cases} 1 \text{ if } A \subseteq B \\ 0 \text{ else} \end{cases}$
$\phi_m(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ else} \end{cases}$
$\phi_{kr}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert B \rvert}$
$\phi_{kp}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \rvert}$
$\phi_j(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}$

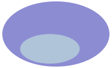
Main consistency indexes



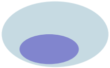
$A \cap B = \emptyset$



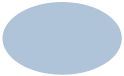
$\phi_p(A,B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{else} \end{cases}$



$\phi_b(A,B) = \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{else} \end{cases}$

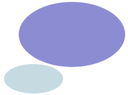


$\phi_q(A,B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{else} \end{cases}$

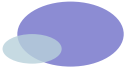


$\phi_m(A,B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{else} \end{cases}$

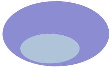
Main consistency indexes



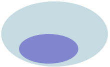
$$A \cap B = \emptyset$$



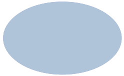
$$\phi_p(A,B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{else} \end{cases}$$



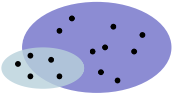
$$\phi_b(A,B) = \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{else} \end{cases}$$



$$\phi_q(A,B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{else} \end{cases}$$



$$\phi_m(A,B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{else} \end{cases}$$



$$\phi_j(A,B) = \frac{|A \cap B|}{|A \cup B|}$$

$$\phi_{kr}(A,B) = \frac{|A \cap B|}{|A|}$$

$$\phi_{kp}(A,B) = \frac{|A \cap B|}{|B|}$$

Uncertainty functions

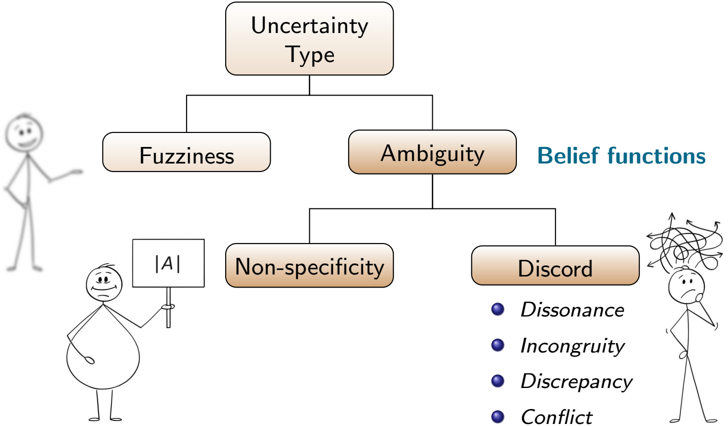
Uncertainty functions can be written under the form:

$$f(A) = \sum_{B \subseteq \mathcal{X}} m(B)\phi(A,B)$$

$\phi(A,B)$	Uncertainty functions		
$\phi_p(A,B) = \begin{cases} 1 \text{ if } A \cap B \neq \emptyset \\ 0 \text{ else} \end{cases}$	Plausibility	$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_p(A,B)$
$\phi_b(A,B) = \begin{cases} 1 \text{ if } B \subseteq A \\ 0 \text{ else} \end{cases}$	Belief	$Bel(A) = \sum_{B \subseteq A} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_b(A,B)$
$\phi_q(A,B) = \begin{cases} 1 \text{ if } A \subseteq B \\ 0 \text{ else} \end{cases}$	Commonality	$q(A) = \sum_{A \subseteq B} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_q(A,B)$
$\phi_m(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ else} \end{cases}$	Mass	$m(A)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_m(A,B)$
$\phi_{kr}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert B \rvert}$	Pignistic probability	$Betp(A) = \sum_{B \subseteq \mathcal{X}} m(B)\frac{\lvert A \cap B \rvert}{\lvert B \rvert}$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_{kr}(A,B)$
$\phi_{kp}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \rvert}$	-	-	
$\phi_j(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}$	-	-	

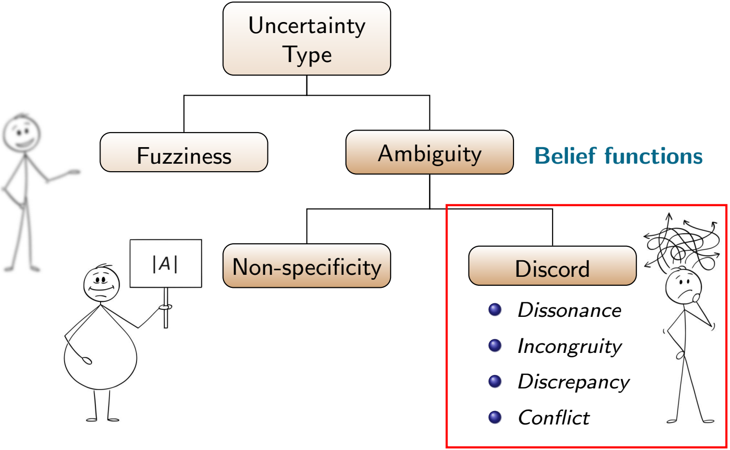
Outline

Several types of uncertainty



Klir and Yuan's typology of uncertainty [8]

Several types of uncertainty



Klir and Yuan's typology of uncertainty [8]

Outline

Several approaches to measuring internal inconsistency I

Entropy-like

Author(s)	Name	Definition	Consistency
Höhle [9]	Confusion	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Bel(A)$	ϕ_b
Yager [10]	Dissonance	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Pl(A)$	ϕ_p
Nguyen [11]	Entropy of random set	$-\sum_{A\subseteq\mathcal{X}} m(A)\log m(A)$	ϕ_m
Klir & Ramer [12]	Discord	$-\sum_{A\subseteq\mathcal{X}} m(A)\log \sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$	ϕ_{kr}
Klir & Parviz [13]	Strife	$-\sum_{A\subseteq\mathcal{X}} m(A)\log \sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$	ϕ_{kp}
Dubois & Prade [14]	Confusion of $\overline{m}$	$-\sum_{A\subseteq\mathcal{X}} m(A)\log q(A)$	ϕ_q

$$\delta(m) = \sum_{A\subseteq\mathcal{X}} m(A)\left(-\log\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right)$$

- Degenerate to Shannon and Hartley entropies
- 0 for $m(\{x\}) = 1$ (max. consistency)

Several approaches to measuring internal inconsistency II

Consistency-based

Author(s)	Name	Definition
Yager [15][†]	Consistency	$1 - \sum_{A \subseteq \mathcal{X}} m(A) Pl(A)$
George & Pal [16]	Total conflict	$\sum_{A \subseteq \mathcal{X}} m(A) \sum_{B \subseteq \mathcal{X}} m(B) \left(1 - \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}\right)$
Daniel [17]	Logical inconsistency	$1 - \max_{x \in \mathcal{X}} Pl(\{x\})$
Destercke & Burger [6]	Probabilistic inconsistency[‡]	$m(\emptyset) = 1 - \max_{A \subseteq \mathcal{X}} Pl(A)$

[†] Yager actually defined the corresponding consistency measure
[‡] The name is derived from Destercke & Burger [6]

$$\boxed{\delta(m) = 1 - \phi(m)}$$

- Do not extend Shannon and Hartley entropies
- 0 if all focal sets intersect

Several approaches to measuring internal inconsistency III

Can we establish links between these measures?

$$\delta(m) = \sum_{A\subseteq\mathcal{X}} m(A)\left(-\log\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right) \qquad \delta(m) = 1-\phi(m)$$

Two steps:

1. Total consistency of m, $\phi(m)$
2. Total inconsistency of m

Outline

Outline

Consistency measures properties

Consistency measure for a mass function

A consistency measure ϕ should satisfy the following properties:

- (cs1) Bounded: $\phi_{\text{min}} \leq \phi(m) \leq \phi_{\text{max}}$
- (cs2) Extreme consistent values:

$$\phi(m) = \phi_{\text{min}} \iff m \text{ totally inconsistent} \iff m(\emptyset) = 1$$
$$\phi(m) = \phi_{\text{max}} \iff m \text{ totally consistent} \implies \ldots$$

- Classically, $\phi_{\text{min}} = 0$ and $\phi_{\text{max}} = 1$
- One definition of total inconsistency
- Several definitions of total consistency

CS GROUP

Consistency measures properties

Consistency measure for a mass function

A consistency measure ϕ should satisfy the following properties:

- (cs1) Bounded: $\phi_{\text{min}} \leq \phi(m) \leq \phi_{\text{max}}$
- (cs2) Extreme consistent values:

$$\phi(m) = \phi_{\text{min}} \iff m \text{ totally inconsistent} \iff m(\emptyset) = 1$$
$$\phi(m) = \phi_{\text{max}} \iff m \text{ totally consistent} \implies \dots$$

- Classically, $\phi_{\text{min}} = 0$ and $\phi_{\text{max}} = 1$
- One definition of total inconsistency
- Several definitions of total consistency

- Probabilistic consistency [Destercke & Burger, 2013 [6]]
- Pairwise consistency [Yager, 1992 [15]]
- Logical consistency [Destercke & Burger, 2013 [6]]

N-consistency of a mass function

Definition (N-consistency)

A mass function m is said to be $\underline{N\text{-consistent}}$, with $1 \leq N \leq |\mathcal{F}|$, iff $\forall \{A_n\}_{n=1}^{N} \subseteq \mathcal{F}$, we have

$$\bigcap_{n=1,\dots,N} A_n \neq \emptyset$$

- Probabilistic consistency coincides with the 1-consistency

$$\forall A \in \mathcal{F}, A \neq \emptyset$$

- Pairwise consistency coincides with the 2-consistency

$$\forall (A, B) \in \mathcal{F}^2, A \cap B \neq \emptyset$$

- Logical consistency coincides with the $|\mathcal{F}|$-consistency

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset$$

A family of consistency measures

Definition (N-consistency measure)

The N-consistency of a mass function m defined over $\mathcal{X}$ is, for $1 \leq N \leq |\mathcal{F}|$, defined by

$$\phi^{(N)}(m) = 1 - m^{(N)}(\emptyset)$$

where $m^{(N)} = m^{(N-1)} \textcircled{\cap} m$ is the conjunctive combination of m with itself N times, with $m^{(0)} = m_{\mathcal{X}}$ the vacuous mass function.

- Measures $\phi^{(N)}$ satisfy (cs1) and (cs2) according to the definition of N-consistency
- The family $\phi^{(N)}$ is ordered $\phi^{(1)}(m) \geq \phi^{(2)}(m) \geq \ldots \geq \phi^{|\mathcal{F}|}$
  - $\phi^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A)$ Probabilistic consistency [Destercke & Burger, 2013]
  - $\phi^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)Pl(A)$ Pairwise consistency [Yager, 1992]
- $\phi^{|\mathcal{F}|}$ is an alternative measure of logical consistency to $\phi_\pi(m) = \max_{x \in \mathcal{X}} Pl(\{x\})$

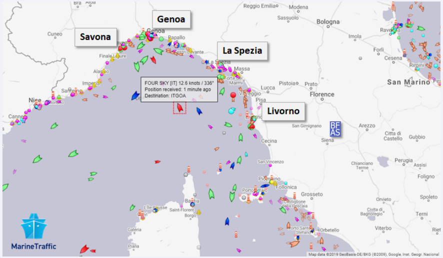
Vessel destination prediction: $\mathcal{X} = \{x_1, \ldots, x_4\}$

S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$



Which one among m_2 and m_3 is more consistent?

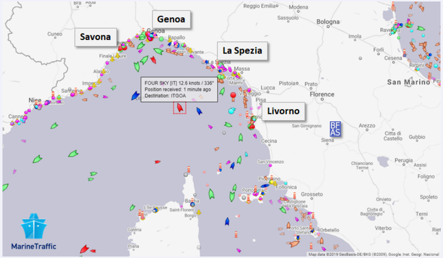
Vessel destination prediction: $\mathcal{X} = \{x_1, \ldots, x_4\}$

S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$



Which one among m_2 and m_3 is more consistent?

	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$
m_2	1	0.92	0.88	0.8
m_3	1	0.66	0.51	0.8

Example: Discrimination and monotonicity

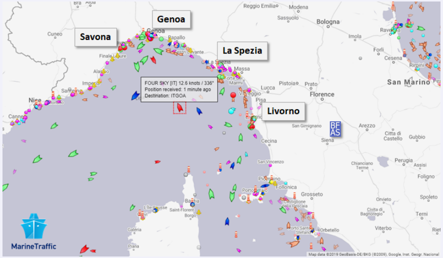
Vessel destination prediction: $\mathcal{X} = \{x_1, \ldots, x_4\}$

S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$



Which one among m_2 and m_3 is more consistent?

	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$
m_2	1	0.92	0.88	0.8
m_3	1	0.66	0.51	0.8

1. m_2 and m_3 are equally consistent according to $\phi^{(1)}$ and ϕ_π
2. They can be discriminated thanks to $\phi^{(2)}$: $m_2 \succ_c m_3$

Example: Discrimination and monotonicity

Vessel destination prediction: $\mathcal{X} = \{x_1, \dots, x_4\}$

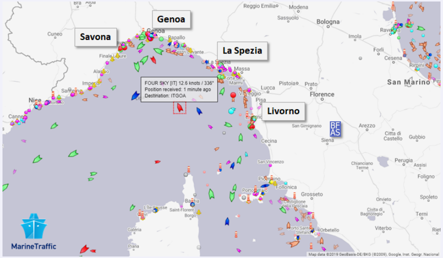
S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$

Which one among m_2 and m_3 is more consistent?



	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$
m_2	1	0.92	0.88	0.8
m_3	1	0.66	0.51	0.8

1. m_2 and m_3 are equally consistent according to $\phi^{(1)}$ and ϕ_π
2. They can be discriminated thanks to $\phi^{(2)}$: $m_2 \succ_c m_3$
3. ϕ_π does not belong to the family $\phi^{(N)}$

Several shades of consistency

Definition (Monotonic N-consistency measure)

The monotonic N-consistency of a mass function m defined over $\mathcal{X}$ is, for $N > 0$, defined by

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

where $m^{(N)} = m^{(N-1)} \textcircled{\scriptsize $\cap$} m$, with $m^{(0)} = m_{\mathcal{X}}$.

- For every mass function m defined over $\mathcal{X}$ with $|\mathcal{F}|$ focal sets:

$$\phi^{(1)}(m) = \phi'^{(1)}(m) \geq \phi'^{(2)}(m) \geq \ldots \geq \phi'^{|\mathcal{F}|}(m) \geq \phi_\pi(m) = \lim_{N \to \infty} \phi'^{(N)}(m)$$

- Measures $\phi'^{(N)}$ satisfy properties (cs1) and (cs2)
- The family $\phi'^{(N)}$ is bounded by the measures of probabilistic and logical consistency
- $\phi'^{|\mathcal{F}|}$ is an alternative measure of logical consistency to ϕ_π

Example: Monotonic consistency of destination predictions

S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$

$$1 - m^{(N)}(\emptyset)$$

$$\left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$	$\phi'^{(1)}(m)$	$\phi'^{(2)}(m)$	$\phi'^{\lvert\mathcal{F}\rvert}(m)$	$\phi'^{(\infty)}(m)$
m_2	1	0.92	0.88	0.8	1	0.96	0.958	0.8
m_3	1	0.66	0.51	0.8	1	0.812	0.801	0.8

- ϕ_π **belongs to the family** $\phi'^{(N)}$:

$$\phi_\pi = \phi'^{(\infty)}$$

Outline

Extending binary consistency indexes

Consistency between N sets

A consistency index between N sets satisfies:

$$\phi^{(N)}(A_1, \ldots, A_N) = \begin{cases} 1 \text{ if } A_1 = \ldots = A_N \\ 0 \text{ if } \bigcap_{i=1,\ldots,N} A_i = \emptyset \end{cases}$$

- For $\phi = \phi_p$:

$$\phi_p^{(N)}(A_1, \ldots, A_N) = \begin{cases} 1 \text{ if } \bigcap_{i=1,\ldots,N} A_i \neq \emptyset \\ 0 \text{ else} \end{cases}$$

- For Bayesian mass functions, the N-wise comparison of focal sets reduces to pair-wise comparison
- It is not true in the general case

Total consistency of a mass function

Consistency of A relatively to m

The consistency of A relatively to a specific set m of $\mathcal{X}$ is defined by:

$$\phi^{(N)}(A|m) = \sum_{B_1 \subseteq \mathcal{X}} m(B_1) \dots \sum_{B_{N-1} \subseteq \mathcal{X}} m(B_{N-1}) \phi^{(N)}(A, B_1, \dots, B_{N-1})$$

Total consistency of a mass function

Consistency of A relatively to m

The consistency of A relatively to a specific set m of $\mathcal{X}$ is defined by:

$$\phi^{(N)}(A|m) = \sum_{B_1 \subseteq \mathcal{X}} m(B_1) \dots \sum_{B_{N-1} \subseteq \mathcal{X}} m(B_{N-1}) \phi^{(N)}(A, B_1, \dots, B_{N-1})$$

- Consistency index for one set: $\phi^{(1)}(A|m) = \phi^{(1)}(A) = \begin{cases} 1 \text{ if } A \neq \emptyset \\ 0 \text{ if } A = \emptyset \end{cases}$
- For $N = 2$ and different ϕ we get the uncertainty functions:

$$\phi(A|m) = \sum_{B \subseteq \mathcal{X}} m(B) \phi(A, B)$$

Total consistency of a mass function

Consistency of A relatively to m

The consistency of A relatively to a specific set m of $\mathcal{X}$ is defined by:

$$\phi^{(N)}(A|m) = \sum_{B_1 \subseteq \mathcal{X}} m(B_1) \dots \sum_{B_{N-1} \subseteq \mathcal{X}} m(B_{N-1}) \phi^{(N)}(A, B_1, \dots, B_{N-1})$$

- Consistency index for one set: $\phi^{(1)}(A|m) = \phi^{(1)}(A) = \begin{cases} 1 \text{ if } A \neq \emptyset \\ 0 \text{ if } A = \emptyset \end{cases}$
- For $N = 2$ and different ϕ we get the uncertainty functions:

$$\phi(A|m) = \sum_{B \subseteq \mathcal{X}} m(B) \phi(A, B)$$

The total consistency of m is then:

$$\phi^{(N)}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \phi^{(N)}(A|m)$$

General formulation of inconsistency and entropy I

Total inconsistency measure

The total inconsistency of a mass function m can be written as:

$$\delta_a^{(N)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)\psi_a\left(\phi^{(N)}(A|m)\right)$$

- $\sum_{A\subseteq\mathcal{X}} m(A)$ is the expectation operator over the focal sets
- ψ_a is the power function:

$$\psi_a(x) = \begin{cases} \frac{1}{a-1}\left(1 - x^{a-1}\right), \text{ if } a \neq 1 \\ -\log(x) \text{ if } a = 1 \end{cases} \qquad \begin{cases} \psi_2(x) = 1 - x \\ \psi_1(x) = -\log(x) \end{cases}$$

- a controls the decrease of the inconsistency as a function of the consistency
- N controls how many sets are compared for measuring the consistency

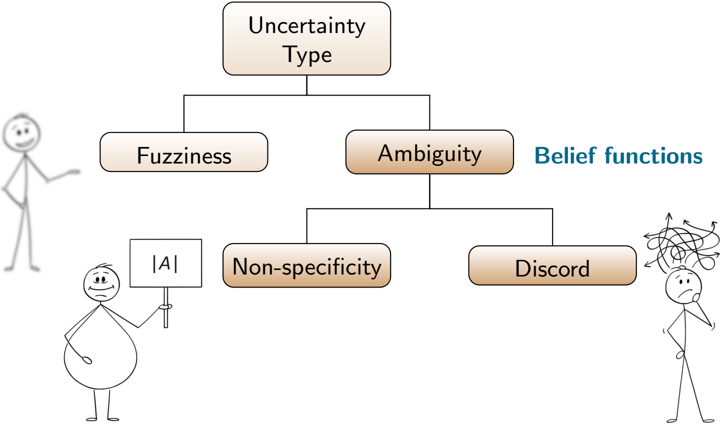
General formulation of inconsistency and entropy II

The choice of the measure reduces to 3 parameters a, N and ϕ

	$a=1$	$a=2$		
	$N=2$		$N=1$	N
ϕ_p	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Pl(A)$	$1-\sum_{A\subseteq\mathcal{X}} m(A)Pl(A)$	$m(\emptyset)=1-\max_{A\subseteq\mathcal{X}} Pl(A)$	$m^{(N)}(\emptyset)$
ϕ_b	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Bel(A)$			
ϕ_q	$-\sum_{A\subseteq\mathcal{X}} m(A)\log q(A)$			
ϕ_m	$-\sum_{A\subseteq\mathcal{X}} m(A)\log m(A)$			
ϕ_{kr}	$-\sum_{A\subseteq\mathcal{X}} m(A)\log\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$			
ϕ_{kp}	$-\sum_{A\subseteq\mathcal{X}} m(A)\log\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$			
ϕ_j		$1-\sum_{A\subseteq\mathcal{X}} m(A)\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}$		

Outline

Several types of uncertainty



Klir and Yuan's typology of uncertainty [8]

Non-specificity and total uncertainty

Non-specificity

$$\delta_{ns}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \log |A|$$

[Dubois & Prade, 1985 [19]]

- $\delta_{ns}(p) = 0$
- $\delta_{ns}(A) = \log |A|$ (Hartley measure)
- Unique measure satisfying properties of *additivity, subadditivity, continuity, branching, normalization and monotonicity*

Non-specificity and total uncertainty

Non-specificity

$$\delta_{ns}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \log |A|$$

[Dubois & Prade, 1985 [19]]

- $\delta_{ns}(p) = 0$
- $\delta_{ns}(A) = \log |A|$ (Hartley measure)
- Unique measure satisfying properties of *additivity, subadditivity, continuity, branching, normalization and monotonicity*

Total uncertainty

Probability transformations

$$\delta_p(m) = -\sum_{x \in \mathcal{X}} p_m(x) \log p_m(x)$$

- $\delta_p(p) = Sh(p)$ (Shannon entropy)
- $\delta_p(A) = H(A)$ (Hartley measure)

Non-specificity and total uncertainty

Non-specificity

$$\delta_{ns}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \log |A|$$

[Dubois & Prade, 1985 [19]]

- $\delta_{ns}(p) = 0$
- $\delta_{ns}(A) = \log |A|$ (Hartley measure)
- Unique measure satisfying properties of *additivity, subadditivity, continuity, branching, normalization and monotonicity*

Total uncertainty

Probability transformations

$$\delta_p(m) = -\sum_{x \in \mathcal{X}} p_m(x) \log p_m(x)$$

- $\delta_p(p) = Sh(p)$ (Shannon entropy)
- $\delta_p(A) = H(A)$ (Hartley measure)

Weighted sum

$$\delta_{a,b}(m) = a\delta(m) + b\delta_{ns}(m)$$

Non-specificity and total uncertainty

Non-specificity

$$\delta_{ns}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \log |A|$$

[Dubois & Prade, 1985 [19]]

- $\delta_{ns}(p) = 0$
- $\delta_{ns}(A) = \log |A|$ (Hartley measure)
- Unique measure satisfying properties of *additivity, subadditivity, continuity, branching, normalization and monotonicity*

Total uncertainty

Probability transformations

$$\delta_p(m) = -\sum_{x \in \mathcal{X}} p_m(x) \log p_m(x)$$

Weighted sum

$$\delta_{a,b}(m) = a\delta(m) + b\delta_{ns}(m)$$

- $\delta_p(p) = Sh(p)$ (Shannon entropy)
- $\delta_p(A) = H(A)$ (Hartley measure)

- δ is either a measure of internal inconsistency or of total uncertainty

Non-specificity and total uncertainty

Non-specificity

$$\delta_{ns}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \log |A|$$

[Dubois & Prade, 1985 [19]]

- $\delta_{ns}(p) = 0$
- $\delta_{ns}(A) = \log |A|$ (Hartley measure)
- Unique measure satisfying properties of *additivity, subadditivity, continuity, branching, normalization and monotonicity*

Total uncertainty

Probability transformations

$$\delta_p(m) = -\sum_{x \in \mathcal{X}} p_m(x) \log p_m(x)$$

Weighted sum

$$\delta_{a,b}(m) = a\delta(m) + b\delta_{ns}(m)$$

Other formulations exist as well . . .

- $\delta_p(p) = Sh(p)$ (Shannon entropy)
- $\delta_p(A) = H(A)$ (Hartley measure)

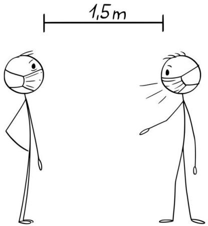
- δ is either a measure of internal inconsistency or of total uncertainty

Outline

Several notions of external inconsistency

Binary measures

- Conflict
- Angle
  Cross-entropy
  Divergence
- Distance



Outline

Conflict definitions

Definition (Total conflict [Destercke & Burger, 2013 [6]])

Two mass functions m_1 and m_2 are said to be totally conflicting if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, where $\mathcal{C}_i = \cup_{A \in \mathcal{F}_i} A$ denote the disjunction of the focal sets of m_i.

Different definitions characterize the state of **non-conflict**: $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

Conflict definitions

Definition (Total conflict [Destercke & Burger, 2013 [6]])

Two mass functions m_1 and m_2 are said to be totally conflicting if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, where $\mathcal{C}_i = \cup_{A \in \mathcal{F}_i} A$ denote the disjunction of the focal sets of m_i.

Different definitions characterize the state of **non-conflict**: $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

- 1-non-conflict: m_1 and m_2 are 1-non-conflicting iff $\forall A \in \mathcal{F}_{12}, A \neq \emptyset$
- 2-non-conflict: m_1 and m_2 are 2-non-conflicting $\forall (A, B) \in \mathcal{F}_{12}^2, A \cap B \neq \emptyset$
- $\mathcal{F}_{12}$-non-conflict: m_1 and m_2 are $\mathcal{F}_{12}$-non-conflicting $\bigcap_{A \in \mathcal{F}_{12}} A \neq \emptyset$

Conflict measures

Inconsistency-based measure of conflict

The conflict between m_1 and m_2 can be defined as the inconsistency of their conjunctive combination:

$$\kappa(m_1, m_2) = 1 - \phi(m_1 \circledcap m_2),$$

where ϕ is a consistency measure

To each consistency measure previously defined, corresponds a conflict measure:

- Dempster's or probabilistic conflict

$$\kappa_1(m_1, m_2) = 1 - \phi'^{(1)}(m_1 \circledcap m_2) = (m_1 \circledcap m_2)(\emptyset)$$

- Logical conflict

$$\kappa_\pi(m_1, m_2) = 1 - \phi'^{(\infty)}(m_1 \circledcap m_2) = 1 - \max_{x \in \mathcal{X}} Pl_{1 \circledcap 2}(\{x\})$$

Several shades of conflict

If we consider the N-consistency:

Definition (N-conflict measure)

The N-conflict between two mass functions m_1 and m_2 for $N \geq 0$, is defined by:

$$\kappa_N(m_1, m_2) = 1 - \left(1 - (m_1 \textcircled{\cap} m_2)^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

where $m^{(N)}$ denotes the N successive conjunctive combinations of m with itself.

- Monotonically ordered family of conflict measures

$$\kappa_1(m_1, m_2) \leq \kappa_2(m_1, m_2) \leq \ldots \leq \kappa_{|\mathcal{F}_{12}|}(m_1, m_2) \leq \kappa_\pi(m_1, m_2) = \lim_{N \to \infty} \kappa_N(m_1, m_2)$$

- Encompasses existing measures of probabilistic and logical conflict
- Satisfy the desirable properties considering the different definitions of non-conflict

Outline

Observations about distances between belief functions

Example (Vessel destination)

$$\mathcal{X} = \{x_1, x_2, x_3, x_4\} = \{\text{Savona}, \text{Genoa}, \text{La Spezia}, \text{Livorno}\}$$

$$\begin{array}{lcl} m_1(\{x_1, x_2, x_3\}) = 0.8 & \overset{?}{\longleftrightarrow} & m^*(\{x_1, x_2\}) = 0.8 & \overset{?}{\longleftrightarrow} & m_2(\{x_4\}) = 0.8 \\ m_1(\mathcal{X}) = 0.2 & & m^*(\mathcal{X}) = 0.2 & & m_2(\mathcal{X}) = 0.2 \end{array}$$

Which of m_1 and m_2 is closer to m^*?

- Because $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$ and $\{x_1, x_2\} \cap \{x_4\} = \emptyset$, we expect

$$d(m^*, m_1) < d(m^*, m_2)$$

- However, neither m_1 nor m_2 share any focal set with m^* (except $\mathcal{X}$)

$$d_I^{(2)}(m^*, m_1) = d_I^{(2)}(m^*, m_2)$$

- The consistency between focal sets has to be considered in the distance measure

Consistency as a norm

Minkowski family of norms L_α can measure the internal consistency of m:

$$\phi_M^{(\alpha)}(m) = \left(\sum_{A\subseteq\mathcal{X}} f(A)^\alpha\right)^{\frac{1}{\alpha}} = \left(\sum_{A\subseteq\mathcal{X}}\left(\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right)^\alpha\right)^{\frac{1}{\alpha}}$$

- Euclidean norm, L_2:

$$\phi_M^{(2)}(m) = \left(\sum_{A,B,C\subseteq\mathcal{X}} m(B)m(C)\phi(B,A)\phi(A,C)\right)^{\frac{1}{2}}$$

  - Jaccard norm for $\phi = \phi'_j$:

$$\phi_{M,j}^{(2)}(m) = \left(\sum_{A,B\subseteq\mathcal{X}} m(A)m(B)\frac{|A\cap B|}{|A\cup B|}\right)^{\frac{1}{2}}$$

CS GROUP

Consistency as a norm

Minkowski family of norms L_α can measure the internal consistency of m:

$$\phi_M^{(\alpha)}(m) = \left(\sum_{A\subseteq\mathcal{X}} f(A)^\alpha\right)^{\frac{1}{\alpha}} = \left(\sum_{A\subseteq\mathcal{X}}\left(\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right)^\alpha\right)^{\frac{1}{\alpha}}$$

- Euclidean norm, L_2:

$$\phi_M^{(2)}(m) = \left(\sum_{A,B,C\subseteq\mathcal{X}} m(B)m(C)\phi(B,A)\phi(A,C)\right)^{\frac{1}{2}}$$

  - Jaccard norm for $\phi = \phi'_j$:

$$\phi_{M,j}^{(2)}(m) = \left(\sum_{A,B\subseteq\mathcal{X}} m(A)m(B)\frac{|A\cap B|}{|A\cup B|}\right)^{\frac{1}{2}}$$

- Chebyshev norm, L_∞:

$$\phi_M^{(\infty)}(m) = \max_{A\subseteq\mathcal{X}} f(A)$$

  - If $\phi = \phi_p$:

$$\phi_{M,p}^{(\infty)}(m) = \max_{A\subseteq\mathcal{X}} Pl(A) = \phi'^{(1)}(m)$$

The 1-consistency measure coïncides with Chebyshev norm of Pl

Minkowski distances family

Minkowski distances

Minkowski family of distances between m_1 and m_2 is the L_α norm of their difference:

$$d_M^{(\alpha)}(m_1, m_2) = \left(\sum_{A \subseteq \mathcal{X}} \left[\sum_{B \subseteq \mathcal{X}} (m_1 - m_2)(B) \phi(A, B) \right]^{\alpha} \right)^{\frac{1}{\alpha}}, \quad \alpha > 1$$

That we can write under a vector-matrix form:

$$d_M^{(\alpha)}(m_1, m_2) = \left(\left[(\mathbf{\Phi m}_1 - \mathbf{\Phi m}_2)^{\frac{\alpha}{2}} \right]' \left[(\mathbf{\Phi m}_1 - \mathbf{\Phi m}_2)^{\frac{\alpha}{2}} \right] \right)^{\frac{1}{\alpha}}$$

avec

- $\mathbf{\Phi}$ matrix of binary consistency indices
- $\mathbf{m}$ mass function vector

L_α distances

$\phi(A,B)$	L_1	L_2	L_∞
ϕ_m		$\delta^{(2)}_{M,m}$	
ϕ_b	$\delta^{(1)}_{M,b}$	$\delta^{(2)}_{M,b}$	$\delta^{(\infty)}_{M,b}$
ϕ_p	$\delta^{(1)}_{M,p}$	$\delta^{(2)}_{M,p}$	
$\phi_p(x,B)$	$\delta^{(1)}_{M,px}$		$\delta^{(\infty)}_{M,px}$
ϕ_{kr}			$\delta^{(\infty)}_{M,kr}$
$\phi_{krx}(x,B)$		$\delta^{(2)}_{M,krx}$	$\delta^{(\infty)}_{M,kr}$
ϕ_{mf}		$\delta^{(2)}_{M,mf}$	
ϕ_j		$\delta^{(2)}_{M,j}$	
ϕ_S		$\delta^{(2)}_{M,S}$	
ϕ_F		$\delta^{(2)}_{M,F}$	

Main L_α distances

$$d^{(2)}_{M,p}(m_1,m_2) = \left(\sum_{A\subseteq\mathcal{X}} (Pl_1(A) - Pl_2(A))^2\right)^{\frac{1}{2}} = d^{(2)}_{M,b}(m_1,m_2)$$

$$d^{(2)}_{M,krx}(m_1,m_2) = \left(\sum_{x\in\mathcal{X}} (BetP_1(\{x\}) - BetP_2(\{x\}))^2\right)^{\frac{1}{2}}$$

$$d^{(\infty)}_{M,p}(m_1,m_2) = \max_{A\subseteq\mathcal{X}} (Pl_1(A) - Pl_2(A))$$

$$d^{(\infty)}_{M,px}(m_1,m_2) = \max_{x\in\mathcal{X}} (Pl_1(\{x\}) - Pl_2(\{x\}))$$

Euclidean Jaccard distance

Jaccard distance

The Euclidian Jaccard distance between two mass functions m_1 and m_2 is defined by:

$$d_{M,j}^{(2)}(m_1, m_2) = \left(\phi_j^{(2)}(m_1) + \phi_j^{(2)}(m_2) - 2\phi_j^{(2)}(m_1, m_2)\right)^{\frac{1}{2}}$$

- $\phi_j^{(2)}(m_1, m_2)$ is Jaccard inner product:

$$\phi_j^{(2)}(m_1, m_2) = \sum_{A,B \subseteq \mathcal{X}} m_1(A) m_2(B) \frac{|A \cap B|}{|A \cup B|}$$

- $\phi_j(A, B) = \frac{|A \cap B|}{|A \cup B|}$ similarity between A and B
- $d_{M,j}^{(2)}$ is a full metric

Other distances

- **Hellinger distance family** e.g., [Ristic & Smets, 2006 [25]]

$$d^{(H)}(m_1, m_2) = \left(1 - \sum_{A,B \subseteq \mathcal{X}} (m_1(A)m_2(B)\phi(A, B))^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

- **Information-based distances family** e.g., [Denœux, 2001 [26]]

$$d(m_1, m_2) = |\delta(m_1) - \delta(m_2)|$$

  where δ is a unary uncertainty measure
- **Belief-Interval distance** [Han, Dezert, Yang, 2014 [27]]
- **Wasserstein distance** [Bronevich and Rozenberg, 2021 [28]]
- **Distance on ordered sets** [Martin, 2022 [29]]

Inner product and cross-entropy

Reminder: General formulation for $N = 2$

$$\delta_a^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)\psi_a\left(\phi^{(N)}(A|m)\right)$$

Inner product and cross-entropy

Reminder: General formulation for $N = 2$

$$\delta_a^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)\psi_a\left(\phi^{(N)}(A|m)\right)$$

If we use two mass functions m_1 and m_2:

$$\delta_a^{(2)}(m_1, m_2) = \sum_{A \subseteq \mathcal{X}} m_1(A)\psi_a\left(\phi^{(2)}(A|m_2)\right)$$

- For $m_1 = m_2 = m$ and $N = 2$, we retrieve unary measures: $\delta_a^{(2)}(m, m) = \delta_a^{(2)}(m)$
- For $a = 2$:

$$\delta_2^{(2)}(m_1, m_2) = 1 - \sum_{A, B \subseteq \mathcal{X}} m_1(A)m_2(B)\phi(A, B)$$

  - $\phi = \phi_p \longrightarrow$ Dempster's conflict
  - $\phi = \phi_j \longrightarrow$ 1 - Jaccard inner product
  - $\phi = \phi_m \longrightarrow$ Wen's cosinus (unnormalized) Wen *et al.*, 2000 [30]

CS GROUP

Outline

Outline

Consistency as a norm (bis)

Reminder: Monotonic N-consistency measure

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

The state of total inconsistency is such that:

$$m(\emptyset) = 1 \iff Pl(A) = 0, \ \forall A \subseteq \mathcal{X}$$

Distance to total inconsistency

$$\phi'^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A) = \phi^{(\infty)}_{M,p}(m) = d^{(\infty)}_{M,p}(m, m_\emptyset)$$

$$\phi'^{(\infty)}(m) = \max_{x \in \mathcal{X}} Pl(\{x\}) = \phi^{(\infty)}_{M,px}(m) = d^{(\infty)}_{M,px}(m, m_\emptyset))$$

Consistency as a norm (bis)

Reminder: Monotonic N-consistency measure

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

The state of total inconsistency is such that:

$$m(\emptyset) = 1 \iff Pl(A) = 0,\ \forall A \subseteq \mathcal{X}$$

Distance to total inconsistency

$$\phi'^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A) = \phi^{(\infty)}_{M,p}(m) = d^{(\infty)}_{M,p}(m, m_\emptyset)$$

$$\phi'^{(\infty)}(m) = \max_{x \in \mathcal{X}} Pl(\{x\}) = \phi^{(\infty)}_{M,px}(m) = d^{(\infty)}_{M,px}(m, m_\emptyset))$$

The consistency of a mass function can be seen as its distance to the totally inconsistent knowledge state

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state

Conflict and distance

$$\begin{cases} \kappa_1(m_1, m_2) = 1 - d^{(\infty)}_{M,p}(m_1 \textcircled{\cap} m_2, m_\emptyset) \\ \kappa_\pi(m_1, m_2) = 1 - d^{(\infty)}_{M,px}(m_1 \textcircled{\cap} m_2, m_\emptyset) \end{cases}$$

GROUP

Conflict and distance

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state

Conflict and distance

$$\begin{cases} \kappa_1(m_1, m_2) = 1 - d^{(\infty)}_{M,p}(m_1 \textcircled{\cap} m_2, m_\emptyset) \\ \kappa_\pi(m_1, m_2) = 1 - d^{(\infty)}_{M,px}(m_1 \textcircled{\cap} m_2, m_\emptyset) \end{cases}$$

- The Euclidean distance between plausibilities quantifies how much m_1 and m_2 are in conflict with the same sets (according to κ_1)

$$d^{(2)}_{M,p}(m_1, m_2) = \left(\sum_{A \subseteq \mathcal{X}} (\kappa_1(m_1, m_A) - \kappa_1(m_2, m_A))^2 \right)^{\frac{1}{2}}$$

Outline

Combining conflict and distance

Distance and conflict capture different notions

To capture the "total discrepancy" between belief functions:

- **Two-dimensional measures** (*e.g.*, [Liu, 2006 [31]]):

$$\delta^{2D} = \left(m(\emptyset); d_{kr}^{(\infty)}(m_1, m_2)\right)$$

Combining conflict and distance

Distance and conflict capture different notions

To capture the "total discrepancy" between belief functions:

- **Two-dimensional measures** (*e.g.*, [Liu, 2006 [31]]):

$$\delta^{2D} = \left(m(\emptyset); d_{kr}^{(\infty)}(m_1, m_2) \right)$$

- **Product** [Martin, 2012 [7]]:

$$\delta^{\times}(m_1, m_2) = (1 - \mathrm{Inc}(m_1, m_2)).d_{M,j}^{(2)}(m_1, m_2)$$

with $\mathrm{Inc}(m_1, m_2) = \frac{1}{|\mathcal{F}_1|.|\mathcal{F}_2|} \sum_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} \phi_b(A, B)$ is inclusion index between m_1 and m_2

Can be generalized to other pairs of (conflict; distance) measures

Outline

Measures properties

			Distance	Conflict
(δ_1)	**Boundedness**	$\delta_{\min} \leq \delta(m_1, m_2) \leq \delta_{\max}$		$\times$
(δ_1)'	**Positivity**	$0 \leq \delta(m_1, m_2)$	$\times$	$\times$
(δ_2)'	**Extreme min. value**	$\delta_{\min}$ iff m_1 and m_2 minimally distant / in conflict	$\times$	$\times$
(δ_2)"	**Extreme max. value**	$\delta_{\max}$ iff m_1 and m_2 maximally distant / in conflict	$(\times)$	$\times$
(δ_3)	**Symmetry**	$\delta(m_1, m_2) = \delta(m_2, m_1)$	$\times$	$\times$
(δ_4)	**Insensitivity to refinement**	$\delta(m_1, m_2) = \delta(m_{\rho(1)}, m_{\rho(2)})$		$\times$
(δ_5)	**Imprecision monotonicity**	$m_1 \sqsubseteq_s m'_1 \Rightarrow \delta(m_1, m_2) \geq \delta(m'_1, m_2)$		$\times$
(δ_6)	**"Ignorance is bliss"**	$\delta(m_{\mathcal{X}}, m) = 1 - \phi(m)$		$\times$
(δ_7)	**Reflexivity**	$\delta(m, m) = 0$	$\times$	
(δ_8)	**Separability**	$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$	$\times$	
(δ_9)	**Triangle inequality**	$\delta(m_1, m_2) \leq \delta(m_1, m_3) + \delta(m_3, m_2)$	$\times$	

Reflexivity

Reflexivity

$$\delta(m, m) = 0$$

- Generally not satisfied by conflict measures. For instance,

$$(m \bigcirc\!\!\!\!\cap\, m)(\emptyset) \neq 0$$

- Relaxing reflexivity allows to express a notion of "internal conflict" (or internal inconsistency)

Separability

Separability

$$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$$

- Not required for conflict measures
- Satisfied by (full) metric measures
- Not satisfied by pseudo-metric measures

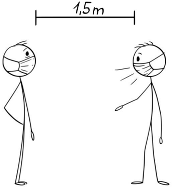
m cannot be more in conflict with $m_{\mathcal{X}}$ (total ignorance) than its internal inconsistency

“Ignorance is bliss”

$$\delta(m, m_{\mathcal{X}}) = 1 - \phi(m)$$

- $\phi(m)$ is a measure of internal inconsistency
- Typically required for conflict measures
- Not required for distance measures

Conflict or distance? Which measure?



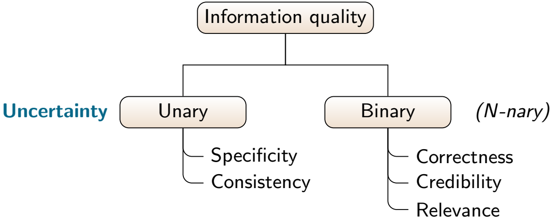
To select the proper measure, we have the following degrees of freedom:

- The desirable properties (separability, reflexivity, "ignorance is bliss", etc) conveying notions of either distance or conflict
- The consistency degree between focal elements and their meaning
- The meaning of the measure (*e.g.*, value of α in Minkowski family, angle versus distance, conflict versus distance, . . .

Outline

1. Preamble
2. Unary measures (internal inconsistency)
3. Binary measures (external inconsistency)
4. Some applications
5. Conclusions

Information quality dimensions



Outline

Correctness

Correctness refers to a reference mass function representing the "truth"

Definition (Correctness assessment)

The correctness of m relatively to a reference m^* is assessed as

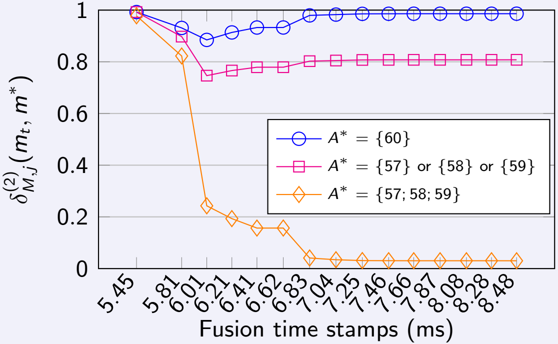
$$\mathrm{Acc}(m) = f\left(\delta(m, m^*)\right)$$

where δ is a binary measure between mass functions, f is a reverse function and m^* represents the true value

- m^* can be categorical, focused on a singleton $m^*(\{x\}) = 1$
- m^* can focus on any other set (m^*_A does not contain any discord (inconsistency), but may contain some non-specificity)
- m^* can be any other mass function

Example: Vessel recognition

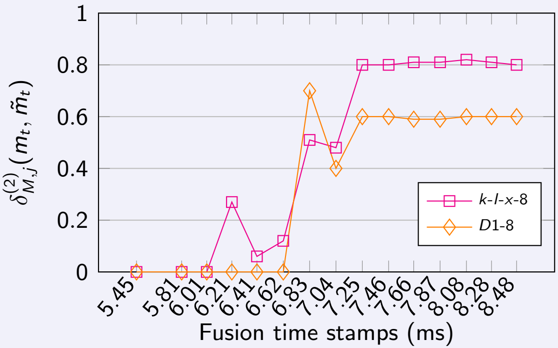
- $\mathcal{X} = \{56, 57, 58, 59, 60\}$ list of vessel IDs
- Four sources, $\{s_i\}$, $i = 1, \ldots, 4$ provide evidence about features which match or not the vessels' features
- Sequential fusion $m_t = m_{t-1} \oplus m_t^{(i)}$
- The correctness is estimated based on the distance of m_t to a m^*
- Convergence toward $A^* = \{57, 58, 59\}$ which is the best answer
- $|A| \neq 1$ denoting some non-specificity
- Elements of A cannot be discriminated



Correctness assessment - *Mass function approximation*

Example: Vessel recognition

- To keep the number of focal sets under control, an approximation algorithm replaces m_t by $\tilde{m}_t$
- A maximum of 8 focal sets are kept
- The correctness of $\tilde{m}_t$ is estimated based on its distance to m_t
- Two approximation algorithms are compared: Tessem's *k-l-x* algorithm and Bauer's *D*1 algorithm
- The *D*1 algorithm provides the closest approximation



Outline

Credibility

Credibility refers here to how much a given mass is consistent within a group of other mass functions (without any access to ground truth)

Definition (Credibility assessment)

The credibility of m relatively to a set of other mass functions m_i is assessed as

$$\mathrm{Cre}(m) = f\left(\delta(m, \{m_i\}_i)\right)$$

where δ is a binary measure between mass functions, $\{m_i\}_i$ is a set of mass functions and f is a reversing function

- The highest $\delta(m, \{m_i\}_i)$ the lower its credibility

Example of credibility measure:

$$\mathrm{Cre}(m) = \left([1 - \delta(m, m_{\oplus})]^{\lambda}\right)^{\frac{1}{\lambda}}$$

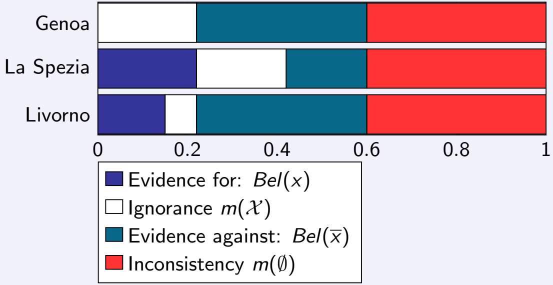
where $m_{\oplus} = \oplus_{i=1, m \neq m_i}^{l} m_i$ and $\lambda > 0$ [Martin *et al.*, 2008 [18]]

Credibility assessment - *Faulty source*

Example: Vessel destination prediction

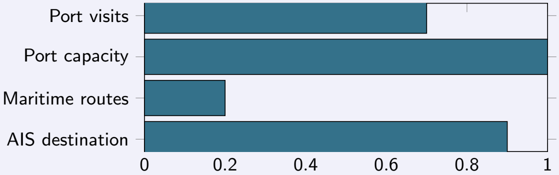
$\mathcal{X} = \{\text{Genoa}, \text{La Spezia}, \text{Livorno}\}$

- Four sources, $\{s_i\}$, $i = 1, \ldots, 4$ provide evidence **for** or **against** destinations
- The inconsistency level suggests some disagreement between sources



Which source provided inconsistent information?

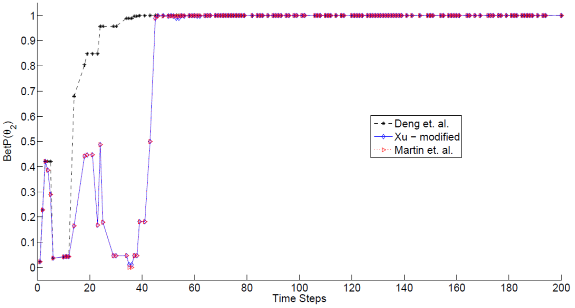
- Credibility of m_i derived from consistency measure (here, Yager)
- Information from `Port capacity` source is maximally credible
- Information from `Maritime routes` source is the least credible



Credibility assessment - *Dynamic discounting*

Example: Vehicle recognition

- Fusion problem with possible missassociation
- Distance measures are used to assess the information credibility
- Non-credible pieces of information are discounted or discarded from the fusion process
- Several credibility measures are compared based on the final pignistic probability of the targeted object



Outline

Relevance

Two notions of relevance:

1. impact of a piece of evidence on previous knowledge
2. relatively to another state of knowledge (retrieval or matching)

Definition (Relevance assessment)

The relevance of m is assessed relatively to another state of knowledge m':

$$\text{Rel}(m) = f\left(\delta(m, m')\right)$$

where δ is a binary measure between mass functions and f is a reversing function

m is relevant if for instance

- $\delta(m, m_0 \bigcirc\!\!\!\!\cap\, m) \neq 0$, m_0 is a previous state of knowledge
- $\delta(m, m_i) \leq \tau$, m_i is a stored knowledge item
- $\delta(m, m_i)$ minimum, m_i is a stored knowledge item

Relevance assessment - *Explaining complex models*

Example: Threat assessment

Source (*i*)	Reliab.	Evidence
AIS	High	AIS received
Classifier "Type"	Medium	Research vessel
AIS "Type"	High	Highly capable
AIS kinematics	Medium	Loitering, Stopped
Intelligence	Medium	ID ''Assumed Friend''
AIS analyzer	High	Inconsistent AIS

(Example of reports and sources)

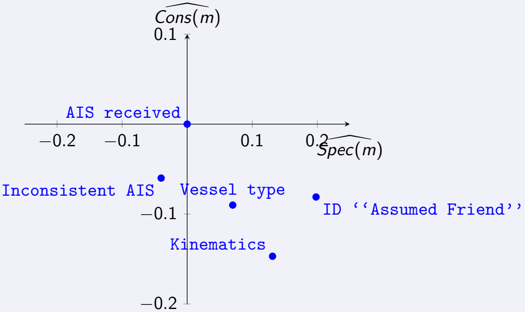
Relevance assessment - *Explaining complex models*

Example: Threat assessment

Source (i)	Reliab.	Evidence
AIS	High	AIS received
Classifier "Type"	Medium	Research vessel
AIS "Type"	High	Highly capable
AIS kinematics	Medium	Loitering, Stopped
Intelligence	Medium	ID ''Assumed Friend''
AIS analyzer	High	Inconsistent AIS

(Example of reports and sources)

$$\mathcal{X} = \{Threat; \neg Threat\}$$



- Global impact of each report on final m
- Specificity and consistency measures

Relevance assessment - *Explaining complex models*

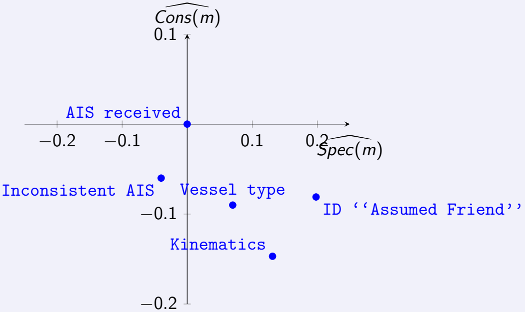
Example: Threat assessment

Source (i)	Reliab.	Evidence
AIS	High	AIS received
Classifier "Type"	Medium	Research vessel
AIS "Type"	High	Highly capable
AIS kinematics	Medium	Loitering, Stopped
Intelligence	Medium	ID ''Assumed Friend''
AIS analyzer	High	Inconsistent AIS

(Example of reports and sources)

- Inconsistent AIS report increases the global inconsistency and reduces the specificity
- ID ''Assumed Friend'' report also increases the global inconsistency but increases the specificity (more informational content)

$$\mathcal{X} = \{Threat; \neg Threat\}$$



- Global impact of each report on final m
- Specificity and consistency measures

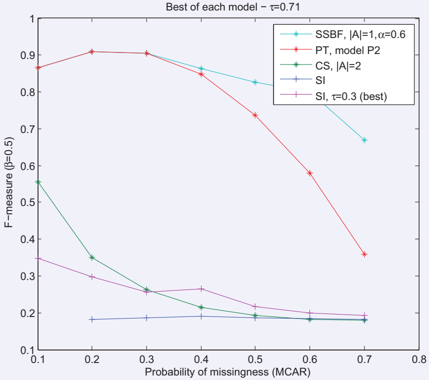
Example: Uncertainty representation for IR with missing data

Problem: Retrieve the set of relevant items r_q to a given query q from an evidential database

$$r_q = \{\mathbf{z} \in \mathcal{Z} | \delta(\mathbf{z}, q) \leq \tau\}$$

where τ is a threshold

- Missing data are replaced by belief functions with different models (classical sets, probabilities, belief functions)
- $\delta = d^{(2)}_{M,j}$ is the Jaccard metric
- Probability of missingness is varied and F-score is assessed
- The best model is a simple support belief function focused on a singleton

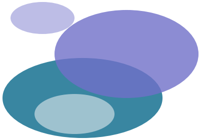


Outline

Conclusions I

Consistency

- A notion of consistency is at the heart of many measures of uncertainty, both unary and binary
  - Consistency between N sets (definition and indexes)
  - Internal consistency of a mass function
  - Consistency as a norm: Distance to total inconsistency $m(\emptyset) = 1$
  - Reversed into inconsistency through ψ_a function



Conclusions II

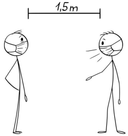
Internal inconsistency

- Two notions:
  - distribution of masses over focal sets
  - interaction between focal sets
- General formulations reduce the choice of the measure to a small number of parameters
  - N, the number focal sets in the interaction
  - ϕ, the consistency index between sets
  - a, the decreasing parameter of the reversing function
- Monotonic family of measures $\phi'^{(N)}(m)$

Conclusions III

External inconsistency

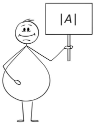
- Family of conflict measures with existing measures as lower and upper bounds
- Distance and conflict measure different notions of discrepancy between belief functions
  - Conflict is defined as the inconsistency resulting from their conjunctive combination
  - Distance is defined as the consistency (norm) of their difference
- Links between some conflict and distance measures
- Other measures: cross-entropy, “angle”, divergence



Conclusions IV

In practice

- Choice of the measure to be guided by semantics and properties
- Distinction between measures of (1) non-specificity, (2) (internal) inconsistency and (3) total uncertainty



- Binary measures allow to quantify notions such as "correctness", "credibility" or "relevance"
- Criteria to refine the fusion process
- Measures of performance, information retrieval, pattern matching, explanations, loss functions in classifiers, ...

References I

[1] F. Pichon, A.-L. Jousselme, and N. BenAbdallah. Several shades of conflict. Fuzzy Sets and Systems, 366:63–84, July 2019.

[2] N. Ben Abdallah, S. Destercke, A.-L.Jousselme, and F. Pichon. An ordered family of consistency measures of belief functions. In S. Destercke, T. Denoeux, F. Cuzzolin, and A. Martin, editors, Proc. of the Int. Conf. on Belief Functions (BELIEF 2018), volume 11069 of Lecture Notes in Computer Science, pages 199–207, Compiègne (France), 2018.

[3] A.-L. Jousselme, F. Pichon, N. BenAbdallah, and S. Destercke. A note about entropy and inconsistency in evidence theory. In Liu Z. Pichon F. (eds) Denoeux T., Lefèvre E., editor, Proc. of the Int. Conf. on Belief Functions (BELIEF 2021), volume 12915 of Lecture Notes in Computer Science, pages 207–214, Shanghai (China), 2021. Springer, Cham.

[4] A.-L. Jousselme and P. Maupin. Distances in evidence theory: Comprehensive survey and generalizations. International Journal of Approximate Reasoning, 53(2):118–145, February 2012.

[5] A.-L. Jousselme, D. Grenier, and É. Bossé. A new distance between two bodies of evidence. Information Fusion, 2(2):91–101, June 2001.

[6] S. Destercke and T. Burger. Toward an axiomatic definition of conflict between belief functions. IEEE Trans. of Systems, Man and Cybernetics - Part B: Cybernetics, 42(2):585–596, 2013.

[7] A. Martin. About conflict in the theory of belief functions. In Advances in Intelligent and Soft Computing 164, Proc. of the 4th Int. Conf. on Belief Functions, 2012.

References II

[8] G. J. Klir and B. Yuan. Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall International, Upper Saddle River, NJ, 1995.

[9] U. Höhle. Entropy with respect to plausibility measures. In Proceedings of the 12^{th} IEEE International Symposium on Multiple Valued Logic, pages 167–169, Paris, 1982.

[10] R. R. Yager. Entropy and specificity in a mathematical theory of evidence. International Journal of General Systems, 9:249–260, 1983.

[11] N. T. Nguyen. On entropy of random sets and possibility distributions. In J. C. Bezdek, editor, The Analysis of Fuzzy Information, volume 1. CRC Press, Boca Raton, FL, 1986.

[12] G. J. Klir and A. Ramer. Uncertainty in the Dempster-Shafer Theory: A critical re-examination. International Journal of General Systems, 18(2):155–166, 1990.

[13] G. J. Klir and B. Parviz. A note on the measure of discord. In D. Dubois, editor, Proceeding of Eighth Conference on Artificial Intelligence, pages 138–141, California, 1992.

[14] D. Dubois and H. Prade. Properties of measures of information in evidence and possibility theories. Fuzzy Sets and Systems, 100 Supplement:35–49, 1999.

[15] R. R. Yager. On considerations of credibility of evidence. International Journal of Approximate Reasoning, 7(1/2):45–72, Aug./Sep. 1992.

[16] T. George and N. R. Pal. Quantification of conflict in Dempster-Shafer framework: A new approach. International Journal of General Systems, 24(4):407–423, 1996.

References III

[17] M. Daniel. Conflicts within and between belief functions. In Eyke Hüllermeier, Rudolf Kruse, and Frank Hoffmann, editors, Computational Intelligence for Knowledge-Based Systems Design, 13th International Conference on Information Processing and Management of Uncertainty, IPMU 2010, volume 6178 of Lecture Notes in Computer Science, pages 696–705, Dortmund, Germany, June 2010. Springer.

[18] A. Martin, A.-L. Jousselme, and C. Osswald. Conflict measure for the discounting operation on belief functions. In Proceedings of the 11th Annual Conference on Information Fusion, Cologne, Germany, 2008.

[19] D. Dubois and H. Prade. A note on measures of specificity for fuzzy sets. International Journal of General Systems, 10:279–283, 1985.

[20] A. Ramer. Uniqueness of information measure in the theory of evidence. Fuzzy Sets and Systems, 24(2):183–196, 1987.

[21] Y. Maeda, H. T. Nguyen, and H. Ichihashi. Maximum entropy algorithms for uncertainty measures. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 1(1):69–93, 1993.

[22] D. Harmanec and G. J. Klir. Measuring total uncertainty in Dempster-Shafer theory: A novel approach. International Journal of General Systems, 22(4):405–419, 1994.

[23] G. J. Klir and M. J. Wierman. Uncertainty-Based Information: Elements of Generalized Information Theory, volume 15 of Studies in Fuzziness and Soft Computing. Physica-Verlag, Heidelberg, New York, 2nd edition, 1999.

[24] R. Jiroušek and P. P. Shenoy. A new definition of entropy of belief functions in the Dempster-Shafer theory. International Journal of Approximate Reasoning, 92:49–65, 2018.

[25] R. Jirousek and P. P. Shenoy. On properties of a new decomposable entropy of Dempster-Shafer belief functions. International Journal of Approximate Reasoning, 199:260–279, 2020.

[26] J. Dezert and A. Tchamova. On the effectiveness of measures of uncertainty of basic belief assignments. Information & Security, 52:9–36, 2022.

[27] D. Han, J. Dezert, and Y. Yang New distance measures of evidence based on belief intervals. In Fabio Cuzzolin, editor, Belief Functions: Theory and Applications, pages 432–441, Cham, 2014. Springer International Publishing.

[28] A. G. Bronevich and I. N. Rozenberg. The measurement of relations on belief functions based on the kantorovich problem and the wasserstein metric. International Journal of Approximate Reasoning, 131:108–135, 2021.

[29] A. Martin. Belief functions on ordered frames of discernment. In Sylvie Le Hégarat-Mascle, Isabelle Bloch, and Emanuel Aldea, editors, Belief Functions: Theory and Applications, pages 129–138. Springer International Publishing, 2022.

[30] C. Wen, Y. Wang, and X. Xu. Fuzzy information fusion algorithm of fault diagnosis based on similarity measure of evidence. In Advances in Neural Networks, volume 5264 of Lecture Notes in Computer Science, pages 506–515. Springer Berlin / Heidelberg, 2008.

[31] W. Liu. Analyzing the degree of conflict among belief functions. Artificial Intelligence, 170:909–924, 2006.

[32] B. Tessem. Approximations for efficient computation in the theory of evidence. Artificial Intelligence, 61(2):315–329, June 1993.

References V

[33] M. Bauer. Approximation algorithms and decision making in the Dempster-Shafer theory of evidence - An empirical study. International Journal of Approximate Reasoning, 17(2-3):217–237, 1997.

[34] M. C. Florea, A.-L. Jousselme, and É. Bossé. Dynamic estimation of evidence discounting rates based on information credibility. RAIRO Operations Research, 44(4):285–306, October 2011.

[35] P. Kowalski, M. Zocholl, and A.-L. Jousselme. Explaining the impact of source behaviour in evidential reasoning. Information Fusion, SI: Advances in Explainable (XAI) and Responsible (RAI) Artificial Intelligence, 2021. In press.

[36] A.-L. Jousselme and P. Maupin. An evidential pattern matching approach for vehicle identification. In Denoeux and Masson, editors, Advances in Intelligent and Soft Computing 164, Proc. of the 2nd Int. Conf. on Belief Functions, pages 45–52, Compiegne, France, May 2012. Springer-Verlag.

Questions ?

Anne-Laure Jousselme
Anne-Laure.Jousselme@csgroup.eu