Positioning belief functions among uncertainty theories

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Outline

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2. Limited expressiveness of probability
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4. Blending set-valued and probability
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Origins of uncertainty

- The variability of observed natural phenomena : **randomness.**
  - Coins, dice…: what about the outcome of the next throw?
- The lack of information: **incompleteness**
  - because of information is often lacking, knowledge about issues of interest is generally not perfect.
- Conflicting testimonies or reports: **inconsistency**
  - The more sources, the more likely the inconsistency

Example

- **Frequentist**: daily quantity of rain in Siena
  - Represents variability: it may change every day
  - It is objective: can be estimated through statistical data
- **Incomplete information** : Birth date of Brazilian President
  - It is not a variable: it is a constant!
  - Information is subjective: Most may have a rough idea (an interval), a few know precisely, some have no idea.
  - Statistics on birth dates of other presidents do not help much.
- **Inconsistent information** : several conflicting testimonies
  - Too much information
  - cannot deduce "everything follows" like in math…

Knowledge vs. evidence

- There are two kinds of information that help us make decisions in the course of actions:
  - Generic knowledge: about repeatable events
    - pertains to a population of observables (e.g. statistical knowledge)
    - Describes a general trend based on objective data
    - Tainted with exceptions
    - Deals with observed frequencies or ideas of typicality
  - Singular evidence: about unique events
    - Consists of direct information about the current world.
    - pertains to a single situation
    - Can be unreliable, uncertain (e.g. unreliable testimony)

The roles of probability

Probability theory is generally used for representing two notions:

1. **Randomness:** capturing variability through repeated observations.
2. **Belief:** describes a person's opinion on the occurrence of a singular event.

As opposed to frequentist probability, subjective probability that models unreliable evidence is not necessarily related to statistics.

Measuring belief of unique events by probability

- **Counting cases**: find symmetries that justify a uniform distribution on cases (Laplace):
  - Assuming determinism: randomness is a result of ill-known conditions
  - No reason to believe one outcome is more likely than the other (insufficient reason principle, equipossibility)
- **Using statistical probability** for assessing belief (Hacking)
  - Generic knowledge = probability distribution P
  - $\text{belief}_{\text{NOW}}(A) = \text{Freq}_{\text{POPULATION}}(A)$: equating belief and frequency
- **Direct elicitation** as subjective probabilities **of non-repeatable events** with no frequentist flavor
  - frequencies may not be available nor known
  - non repeatable events.

SUBJECTIVE PROBABILITIES
(Bruno de Finetti, 1935)

- A set S of mutually exclusive, exhaustive alternatives (states of nature, affairs, possible worlds).
- P(A) = *belief degree* of an agent (You) on the (next) occurrence of event A
  - measured as the **maximal price You agree** to pay for buying a lottery ticket, with reward 1 € if A occurs
  - You must accept to **sell it at the same price** if requested to.
- Often called *Bayesian probability (subjectivist or personal)*
- De Finetti*: "probability does not exist!"*

SUBJECTIVE PROBABILITIES (Bruno de Finetti, 1935)

- **Why a belief state is a single probability distribution** ($\sum_j p_j = 1$):
  - Assume You **buy** all lottery tickets for {i}, i = 1, m.
  - If state s_j is observed, the gain is $1 - \sum_i p_i$
  - it is $\sum_i p_i - 1$ if you have **sold** the tickets
  - *if* $\sum p_i > 1$ You *always lose money if you buy;*
  - *if* $\sum p_i < 1$ *You always lose money if you sell*

So, You must provide degrees of belief such that $\sum_i p_i = 1$, in order to be rational.

Using a single probability distribution to represent incomplete information is not entirely satisfactory:

The betting behavior setting of Bayesian subjective probability enforces a representation of partial ignorance based on single probability distributions.

1. **Ambiguity :** In the absence of information, how can a uniform distribution tell pure randomness and ignorance apart ?
2. **Instability :** A uniform prior on $x \in [a, b]$ induces a non-uniform prior on $f(x) \in [f(a), f(b)]$ if f is increasing and non-affine: *ignorance produces information*
3. **Empirical refutation**: When information is missing, decision-makers do not always choose according to a single subjective probability (Ellsberg paradox).

Set-Valued Representations of Partial Knowledge

- An ill-known quantity x is represented as a *disjunctive* set, i.e. a subset $E \subseteq S$ of *mutually exclusive values*, one of which is the real one.
- Pieces of information of the form $x \in E$
  - **Intervals** $E = [a, b]$: good for representing incomplete numerical information: birth-date $\in$ [1940-1970]
  - **Classical Logic**: good for representing incomplete symbolic (Boolean) information

E = Models of a wff ϕ stated as true.

*This kind of information is subjective (**epistemic set**)*

SETS and SETS

- Do not mix up
  - **A set-valued variable X**: the set of languages a person can speak (A = {English (and) French})

    a conjunction of values, and a real set.

    X = A is precise (Ontic set)
  - **An ill-known point-valued variable x** : the birth nationality of this person

    (E = {English (exor) French})

    a disjunction of values, and an epistemic set (a piece of information in the head of an agent).

    $x \in E$ is imprecise (epistemic set)

BOOLEAN POSSIBILITY THEORY

Natural set functions under incomplete information*:*

If all we know is that $x \in E$ then

- Event A is possible if $A \cap E \neq \emptyset$ (logical consistency)

  $\Pi(A) = 1$, and 0 otherwise

- Event A is sure (necessary) if $E \subseteq A$ (logical deduction)

  $N(A) = 1$, and 0 otherwise

Characteristic properties

$$N(A) = 1 - \Pi(A^c)$$

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B));$$

$$N(A \cap B) = \min(N(A), N(B)).$$

Used in modal (epistemic and doxastic) logics (Hintikka)

REMARKS

- Uncertainty is ternary
  - A is surely true : $N(A)= 1$ (hence $\Pi(A) = 1$)
  - A is surely false : $N(A^c)= 1$ (i.e., $\Pi(A^c) = 1$)
  - A is unknown : $N(A) = N(A^c)= 0$

    ($\Pi(A) = \Pi(A^c) = 1$)
- $N(A) \leq \Pi(A)$ for all A.
- $E_1 \subseteq E_2$: E_1 is more informative than E_2

(Equivalently: $\Pi1 \leq \Pi2$, or $N1 \geq N2$)

Motivation for going beyond probability and epistemic logic

- **Traditional tools to representing uncertainty are**
  - **Probability distributions :** good for expressing variability, but information demanding
  - **Sets**: good for representing incomplete information, but often crude modal representation of uncertainty
- *Find representations that allow for both aspects of uncertainty*.
- Toward formalisms that distinguish between uncertainty due to variability from uncertainty due to lack of knowledge or missing information.

Find a gradual representation of uncertainty due to incompleteness

- *More expressive and informative than sets and epistemic logic*
- *Less demanding than single probability distributions*
- *Explicitly allows for missing information*

GRADUAL REPRESENTATIONS OF UNCERTAINTY USING CAPACITIES

Belief is a matter of degree !

- **Family of propositions or events $\mathcal{E}$ forming a Boolean Algebra**
  - S, Ø are events that are certain and ever impossible respectively.
- **A capacity** g : a function from $\mathcal{E}$ in [0,1] such that
  - $g(\emptyset) = 0 \quad ; \quad g(S) = 1$
  - if A implies (= included in) B then g(A) **≤** g(B) (monotony)
- **g is a confidence measure:** g(A) quantifies the confidence of an agent in proposition A.

BASIC PROPERTIES OF CONFIDENCE MEASURES

- **Due to monotonicity :**
  - **$g(A \cup B) \geq \max(g(A), g(B))$;**
  - **$g(A \cap B) \leq \min(g(A), g(B))$**
- Capacities include :
  - probability measures : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - possibility measures $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$
  - necessity measures $N(A \cap B) = \min(N(A), N(B))$

- ***The two latter functions do not require a numerical setting***

A GENERAL SETTING FOR REPRESENTING GRADED CERTAINTY AND PLAUSIBILITY

- 2 conjugate set-functions Pl and Cr generalizing probability P, possibility Π, and necessity N.
- **Conventions** : Pl for plausibility; Cr for certainty
  - Pl(A) = 0 "impossible" ; Cr(A) = 1 "certain"
  - Pl(A) =1 ; Cr(A) = 0 "ignorance" **(no information)**
  - Pl(A) – Cr(A) quantifies ignorance about A
- **Postulates**
  - Cr and Pl are monotonic under inclusion (= capacities).
  - Cr(A) ≤ Pl(A) "certain implies plausible"
  - $Pl(A) = 1 - Cr(A^c)$ duality certain/plausible
  - If Pl = Cr then it is P.

Canonical examples of (Cr, Pl) pairs

- Accounting for both variability and incomplete knowledge requires **(Cr, Pl) pairs**
- Obtained by combining probability and sets.
  - **Sets of probabilities**: imprecise probability theory
  - **Random(ised) sets**: Dempster-Shafer belief functions
  - **Fuzzy sets**: numerical possibility theory
- *Each event has a degree of belief (certainty) and a degree of plausibility, instead of a single degree of probability*

Possibility Theory

(Shackle, 1961, Zadeh, 1978)

- *Some values are more plausible than other ones*: A piece of information "$x \in E$" admits of *degrees* of possibility
- E is a (normalized) fuzzy set:
  - μ_E: S $\rightarrow$[0, 1] degrees of membership
  - $\mu_E(s) = 1$ for some s
- **Epistemic fuzzy set** : μ_E is interpreted as possibility distribution

$$\pi_x(s) = \mu_E(s) = \text{Possibility}(x = s)$$

Possibility distibutions

- **Conventions:**
  - $\forall s, \pi_x(s)$ is the degree of plausibility of $x = s$
  - $\pi_x(s) = 0$ iff $x = s$ is impossible, totally surprising
  - $\pi_x(s) = 1$ iff $x = s$ is normal, fully plausible, unsurprising (but no certainty)
- **Consistency** : there is at least one $s \in S$ such that $\pi_x(s) = 1$.

But there can be many s with $\pi_x(s) = 1$

Improving expressivity of incomplete information representations

- *What about the birth date of the president?*
- **partial ignorance with ordinal preferences :** May have reasons to believe that 1953 > 1952 ≡ 1954 > 1951 ≡ 1955 > 1950 > 1956 > 1959
- **Linguistic information** described by fuzzy sets: " he is old " : membership μ_{OLD} is interpreted as a possibility distribution on possible birth dates.

- **Nested intervals** $E_1, E_2, \ldots E_n$ **with confidence levels** $N(E_i) = a_i$:
  - $\pi(x) = \min_{i = 1, \ldots n} \max(\mu_{Ei}(x), 1 - a_i)$

Comparing information states

- **π' at least as specific as π if and only if** $\pi' \leq \pi$

(Any possible value according to π' is at least as possible as according to π : *π' is more informative than π)*

  - COMPLETE KNOWLEDGE: The most specific ones
    For some $s_0 \in S$: $\pi(s_0) = 1$; $\pi(s) = 0$ otherwise
  - IGNORANCE: $\pi(s) = 1, \forall s \in S$

- **Principle of least commitment** (minimal specificity): In a given information state, any value not proved impossible is supposed to be possible : **maximise possibility degrees**!

POSSIBILITY AND NECESSITY OF AN EVENT

How confident are we that x $\in$ A $\subset$ S ? (*an event A occurs*) given a possibility distribution on S

- $\Pi(A) = \max\{\pi(s) : s \in A\}$: plausibility as consistency with π: *to what extent A is consistent with π*

  (= some x $\in$ A is possible)

  The degree of possibility *that x $\in$ A*

- $N(A) = 1 - \Pi(A^c) = \min\{1 - \pi(s) : s \notin A\}$

  certainty as entailment from π

  to what extent no element outside A is possible

  = to what extent π implies A

  The degree of certainty (necessity) that x $\in$ A

Basic properties

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B));$$

$$N(A \cap B) = \min(N(A), N(B)).$$

Mind that most of the time :

$$\Pi(A \cap B) < \min(\Pi(A), \Pi(B));$$

$$N(A \cup B) > \max(N(A), N(B)$$

Example: Total ignorance on A and B = A^c

Corollary $N(A) > 0 \Rightarrow \Pi(A) = 1$

Qualitative vs. quantitative possibility theories

- **Qualitative**:
  - **comparative**: A complete pre-ordering $\geq_\pi$ on U or A well-ordered partition of U: E1 > E2 > … > En
  - **absolute:** $\pi_x(s) \in L$ = finite chain, complete lattice...
- **Quantitative**: $\pi_x(s) \in [0, 1]$, integers...

One must indicate where the numbers come from.

All theories agree on the fundamental maxitivity axiom

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$$

Theories diverge on the conditioning operation

Imprecise probability theory

- A state of information is represented by a family $\mathcal{P}$ of probability distributions over a set X.
  - *For instance: incomplete knowledge of a FREQUENTIST probabilistic model :* $\exists P \in \mathcal{P}$.
- To each event A is attached a probability interval $[P_*(A), P^*(A)]$ such that
  - $P_*(A) = \inf\{P(A), P \in \mathcal{P}\}$
  - $P^*(A) = \sup\{P(A), P \in \mathcal{P}\} = 1 - P_*(A^c)$
- Usually $\mathcal{P}$ is strictly contained in $\{P(A), P \geq P_*\}$ as the latter is convex.

Subjectivist view (Peter Walley)

- $P_{low}(A)$ is the greatest acceptable buying price for a bet on event A winning 1 euro if A occurs
- $P^{high}(A) = 1 - P_{low}(A^c)$ is the least acceptable price for selling this bet.
- $P^{high}(A) > P_{low}(A)$ is allowed (≠ De Finetti)

→ Probability sets of the form $\{P \geq P_{low}\}$

- **Rationality** conditions for buying prices
  - **No sure loss** : $C = \{P \geq P_{low}\}$ not empty
  - **Coherence**: $P_*(A) = \inf\{P(A), P \geq P_{low}\} = P_{low}(A)$
- Not all convex sets of probabilities are of this form

Capacity-based lower probabilities

- *A capacity g is called coherent if and only if*

$$\mathcal{C} = \{P: P(A) \geq g(A)\} \neq \emptyset$$

$$P_*(A) = \inf\{P(A), P \geq g\} = g(A)$$

- *Then g can be used as a certainty function Cr*
- Lower probabilities satisfy super-additivity:
  if $A \cap B = \emptyset$ then $P_*(A) + P_*(B) \leq P_*(A \cup B)$
- the 2-monotony property is stronger

$$g(A) + g(B) \leq g(A \cup B) + g(A \cap B)$$

- *g is then called a convex capacity. It* ***characterizes*** *the convex set* $\mathcal{C}$.

Credal sets

*A convex probability set C is called a **credal set***

- Some convex sets of probabilities are characterized only by lower expectations of gambles, *not just events*.

Gambles f: S → Real line

f(s) > 0: gain   f(s) < 0 : loss

- $E_{low}(f)$ is the greatest acceptable buying price of gamble f : a lower expectation (and $E^{high}(f) = - P_{low}(-f)$)
- **Desirable gambles**: $E_{low}(f) > 0$
- The family C of probability distributions P such that $E_P(f_i) \geq E_{low}(f_i)$ for a set of gambles f_i is ***convex*** if not empty.

Probability intervals

- **Definition**: To each element s_i in S, assign a probability interval $[l_i, u_i]$, $i = 1, \ldots n$.
- $\mathcal{P} = \{P: l_i \le P(\{s_i\}) \le u_i , i = 1, \ldots n\}$ is a credal set.
- A probability interval model L is **coherent** in the sense of Walley if and only if all bounds u_j , l_i are attained :

$$\sum_{j \neq i} l_j + u_i \le 1 \text{ and } 1 \le \sum_{j \neq i} u_j + l_i$$

  - In particular $\sum_j l_j \le 1 \le \sum_j u_j$

  Then $l_i = \inf\{P(\{s_i\}): P \text{ in } \mathcal{P}\}$; $u_i = \sup\{P(\{s_i\}): P \text{ in } \mathcal{P}\}$

- Induced lower probabilities are 2-monotone capacities (De Campos and Moral), NOT BELIEF FUNCTIONS

$$P_*(A) = \max(\sum_{j \text{ in } A} l_j , 1-\sum_{j \text{ not in } A} u_j)$$

Random sets

- A positive weighting of non-empty subsets: mathematically, **a random set $\mathcal{R}$** on S:

$$\sum_{E \subseteq S} m(E) = 1 \; ; \; m(\emptyset) = 0$$

- A probability distribution *m* on the family of non-empty subsets of a set S.
  - m : *mass function : m(E) = Prob($\mathcal{R}$ = E)*.
  - *focal sets* : E with m(E) > 0 :
- ***Interpretation problem****: disjunctive or conjunctive, ontic or epistemic ???*

Basic set functions from random sets

- **Probabilities of containment** :
  - $P(\mathcal{R} \subseteq A) = \sum_{E_i \subseteq A,\, E_i \neq \emptyset} m(E_i)$
  - $P(A \subseteq \mathcal{R}) = \sum_{A \subseteq E_i,\, E_i \neq \emptyset} m(E_i)$
- **Probability of hitting $\mathcal{R}$**:
  - $P(A \cap \mathcal{R} \neq \emptyset) = \sum_{E_i \cap A \neq \emptyset} m(E_i)$

Three views of random sets

- A set-valued standard random variable: *ontic (conjunctive) random set.* **Matheron**
- A epistemic set of random variables: *epistemic uncertainty about a random variable* (statistical) **Dempster**
- A subjective probability over possible epistemic sets : *uncertain evidence* (non-statistical) **Shafer book, Smets**

Example of conjunctive random set

Experiment on linguistic capabilities of people :

- **Question** to **a population** $\Omega = \{1, \ldots, i, \ldots, n\}$ of n persons: which languages can you speak ?
- **Answers** : Subsets in $\mathcal{L}$ = {Basque, Chinese, Dutch, English, French,….}
- m(E) = % people who speak *exactly* all languages in E (and not other ones)
- Prob(x speaks (at least) A) $= \sum\{m(E) : A \subseteq E\} = P(A \subseteq \mathcal{R})$

  = Q(A) : commonality function in belief function theory
- **Example**: « x speaks English » means « at least English »
- *Belief functions are not meaningful here while the commonality makes sense, contrary to the disjunctive epistemic set case.*

Random disjunctive sets

- A focal set is an epistemic state ($x \in E$)
- ($\mathcal{F}$, m) = randomized incomplete information
- m(E) = probability that the most precise description of the available information is of the form "$x \in E$"

  = probability (only knowing "$x \in E$" and nothing else)

  – It is the portion of probability mass hanging over elements of E without being allocated.
- **DO NOT MIX UP m(E) and Prob(E) :**

$$m(E) = Prob(\{E\})$$

Belief and Plausibility from Epistemic Random Sets

- **degree of certainty** (belief) :
  - $\mathrm{Bel}(A) = \sum_{E_i \subseteq A,\, E_i \neq \emptyset} m(E_i) = P(\mathcal{R} \subseteq A)$
  - total mass of information implying the occurrence of A
  - *(probability of provability)*
- **degree of plausibility** : $P(A \cap \mathcal{R} \neq \emptyset)$
  - $\mathrm{Pl}(A) = \sum_{E_i \cap A \neq \emptyset} m(E_i) = 1 - \mathrm{Bel}(A^c) \geq \mathrm{Bel}(A)$
  - total mass of information consistent with A
  - *(probability of consistency)*

PARTICULAR CASES

- INCOMPLETE INFORMATION:

  $m(E) = 1, m(A) = 0, A \neq E$
- *TOTAL IGNORANCE :* $m(S) = 1$:
  - *For all* $A \neq S, \emptyset$, *Bel(A) = 0, Pl(A) = 1*

- PROBABILITY: if $\forall i, E_i$ = singleton $\{s_i\}$ (disjoint focal sets) : *Hence precise + scattered information*
  - Then, *for all* A, Bel(A) = Pl(A) = P(A)

- POSSIBILITY THEORY : the consonant case

  $E_1 \subseteq E_2 \subseteq E_3 \ldots \subseteq E_n$: imprecise and coherent information
  - iff $Pl(A \cup B) = \max(Pl(A), Pl(B))$, possibility measure
  - iff $Bel(A \cap B) = \min(Bel(A), Bel(B))$,

    necessity measure = consonant belief function

Dempster vs. Shafer-Smets

- A disjunctive random set can represent
  - *Imprecise statistical information* (Dempster intuition) m(E) = frequency of imprecise observations of the form E
    - an ill-known random variable.
    - Bel(E) is a lower probability
  - *Uncertain singular evidence* (Shafer, 1976; Smets)

  m(E) = subjective probability that testimony E is reliable.
    - Degrees of belief directly modelled by Bel : no appeal to an underlying probability.

In both cases, a focal set E is a set of mutually exclusive values and does not represent a real set-valued entity

Canonical examples

- **Statistical probabilities** : Frequentist modelling of a collection of incomplete observations (imprecise statistics) : *incomplete generic information* (Dempster)
- **Uncertain singular information:**
  - **Unreliable testimonies** (Shafer's book) : unreliable singular information
  - **Unreliable sensors** (Smets): the quality of the information depends on the ill-known sensor state.

Dempster original model

- Indirect information (induced from a probability space).
- What we know about a random variable x with range S, based on a sample space $(\Omega, \mathcal{A}, P)$ is a multi-mapping Γ from Ω to S (Dempster):
- *The meaning of Γ from Ω to S* :
  - if we observe ω in Ω then all we know is $x(\omega) \in \Gamma(\omega)$
- $m(A) = \sum P(\{\omega: A = \Gamma(\omega)\})$, $A \subseteq \Omega$ (finite case.)
- Then a belief function is a lower probability.

SELECTION FUNCTIONS

- A *selection function* of the multimapping Γ from Ω to S is a random variable x such that $\forall\ \omega$ in Ω, $x(\omega) \in \Gamma(\omega)$
- Each random variable x induces a probability P_x on S from $(\Omega, \mathcal{A}, P)$: $P_x(A) = P(x^{-1}(A))$.
- The set $\mathfrak{P}_\Gamma = \{P_x : \forall\ \omega \text{ in } \Omega,\ x(\omega) \in \Gamma(\omega)\}$ of probabilities induced by Γ is not convex
- $\mathrm{Bel}(A) = \sum_{\Gamma(\omega) \subseteq A} P(\omega)$

$$= \min\{P_x(A) : P_x \in \mathfrak{P}_\Gamma\}$$

Example of generic belief function: imprecise observations in an opinion poll

- **Question** : who is your preferred candidate
  in C = {a, b, c, d, e, f} ???
  - **To a population** $\Omega = \{1, \ldots, i, \ldots, n\}$ of n persons.
  - **Imprecise responses r =** « $x(i) \in E_i$ » **are allowed**
  - No opinion (r = C) ; « left wing » r = {a, b, c} ;
  - « right wing » r = {d, e, f} ;
  - a moderate candidate : r = {c, d}

- **Definition of mass function**: $m(E) = \text{card}(\{i, E_i = E\})/n$

  = Proportion of imprecise responses « $x(i) \in E$ »

  uniform distribution on Ω, $E_i = \Gamma(i)$,

- *The probability that a candidate in subset* $A \subseteq C$ *is elected is imprecise :*

$$\text{Bel}(A) \leq P(A) \leq \text{Pl}(A)$$

- **There is a fuzzy set F of potential winners**:

$$\mu_F(x) = \sum_{x \in E} m(E) = \text{Pl}(\{x\}) \text{ (contour function)}$$

- $\mu_F(x)$ is an upper bound of the probability that x is elected. It gathers responses of those who *did not give up voting* for x
- Bel({x}) gathers responses of those who claim they will vote for x and no one else.

Example of uncertain evidence : Unreliable testimony (SHAFER-SMETS VIEW)

- « John tells me the president is between 60 and 70 years old, but there is some chance (*subjective* probability p) he does not know and makes it up».
  - E =[60, 70]; Prob(Knowing "$x \in E$ =[60, 70]") = 1 – p.
  - With probability p, John invents the info, so *we know nothing (Note that this is different from a lie)*.
- We get a *simple support belief function :*

$$m(E) = 1 - p \quad \text{and} \quad m(S) = p$$

- Equivalent to a possibility distribution
  - $\pi(s) = 1$ if $x \in E$ and $\pi(s) = p$ otherwise.
- *If John may lie* (probability q): $m(E) = (1 - p)(1 - q)$, $m(E^c) = (1 - p)q$.

Random disjunctive sets vs. imprecise probabilities

- The set $\mathcal{P}_{\text{bel}} = \{P \geq \text{Bel}\}$ is coherent: Bel is a special case of lower probability.
- The set of probabilities $\mathfrak{P}_\Gamma$ induced by selection functions (Dempster model) is but a subset of $\mathcal{P}_{\text{bel}}$.
  - In the election example, $\mathfrak{P}_\Gamma$ is finite.
- However $\mathcal{P}_{\text{bel}}$ is the convex hull of $\mathfrak{P}_\Gamma$:

$$\text{Bel}(A) = \inf \{P(A): P_x \in \mathcal{P}_{\text{bel}}\} = \min\{P_x(A): P_x \in \mathfrak{P}_\Gamma\}$$

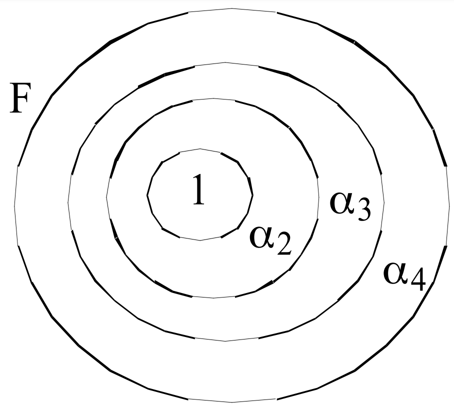
$$\text{Pl}(A) = \sup \{P(A): P_x \in \mathcal{P}_{\text{bel}}\} = \max\{P_x(A): P_x \in \mathfrak{P}_\Gamma\}$$

Random disjunctive sets vs. imprecise probabilities

- Bel is ∞-monotone (super-additive at any order)
  - Order 3: $Bel(A \cup B \cup C) \geq Bel(A) + Bel(B) + Bel(C) - Bel(A \cap B) - Bel(A \cap C) - Bel(B \cap C) + Bel(A \cap B \cap C)$,
  - Order 4: etc..
- For any set function, the solution m to the set of equations $\forall A \subseteq X\ g(A) = \sum_{E_i \subseteq A, E_i \neq \emptyset} m(E_i)$

  is unique (m is the ***Moebius transform*** of g)
  - ***However m is positive iff g is a belief function***

From possibility to belief functions



possibility levels
$1 > \alpha_2 > \alpha_3 > \ldots > \alpha_n$

- Let $m_i = \alpha_i - \alpha_{i+1}$ then $m_1 + \ldots + m_n = 1$,
  with focal sets = cuts $E_i = \{s, \pi(s) \geq \alpha_i\}$
  A consonant basic probability assignment (SHAFER)
- $\pi(s) = \sum_{i:\, s \in Ei} m_i$ (one point-coverage function) = Pl({s}).
- ***Only in the consonant case can m be recalculated from π***
- $Bel(A) = \sum_{Ei \subseteq A} m_i = N(A); Pl(A) = \Pi(A)$

POSSIBILITY AS UPPER PROBABILITY

- Given a numerical possibility distribution π, define

$\mathcal{P}(\pi) = \{P \mid P(A) \leq \Pi(A) \text{ for all } A\}$

  - Then, Π and N can be recovered
    - $\Pi(A) = \sup \{P(A) \mid P \in \mathcal{P}(\pi)\}$;
    - $N(A) = \inf \{P(A) \mid P \in \mathcal{P}(\pi)\}$
  - So π is a faithful representation of a special family of coherent probability measures

- Likewise for belief functions : $\mathcal{P}(m) = \{P \mid P(A) \leq Pl(A), \forall A\}$
    - $Pl(A) = \sup \{P(A) \mid P \in \mathcal{P}(m)\}$;
    - $Bel(A) = \inf \{P(A) \mid P \in \mathcal{P}(m)\}$

From confidence sets to possibility distributions

- Let $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n$ be a nested family of sets
- A set of confidence levels $a_1, a_2, \ldots a_n$ in [0, 1]
- Consider the set of probabilities

$$\mathcal{P} = \{P, P(E_i) \geq a_i, \text{ for } i = 1, \ldots n\}$$

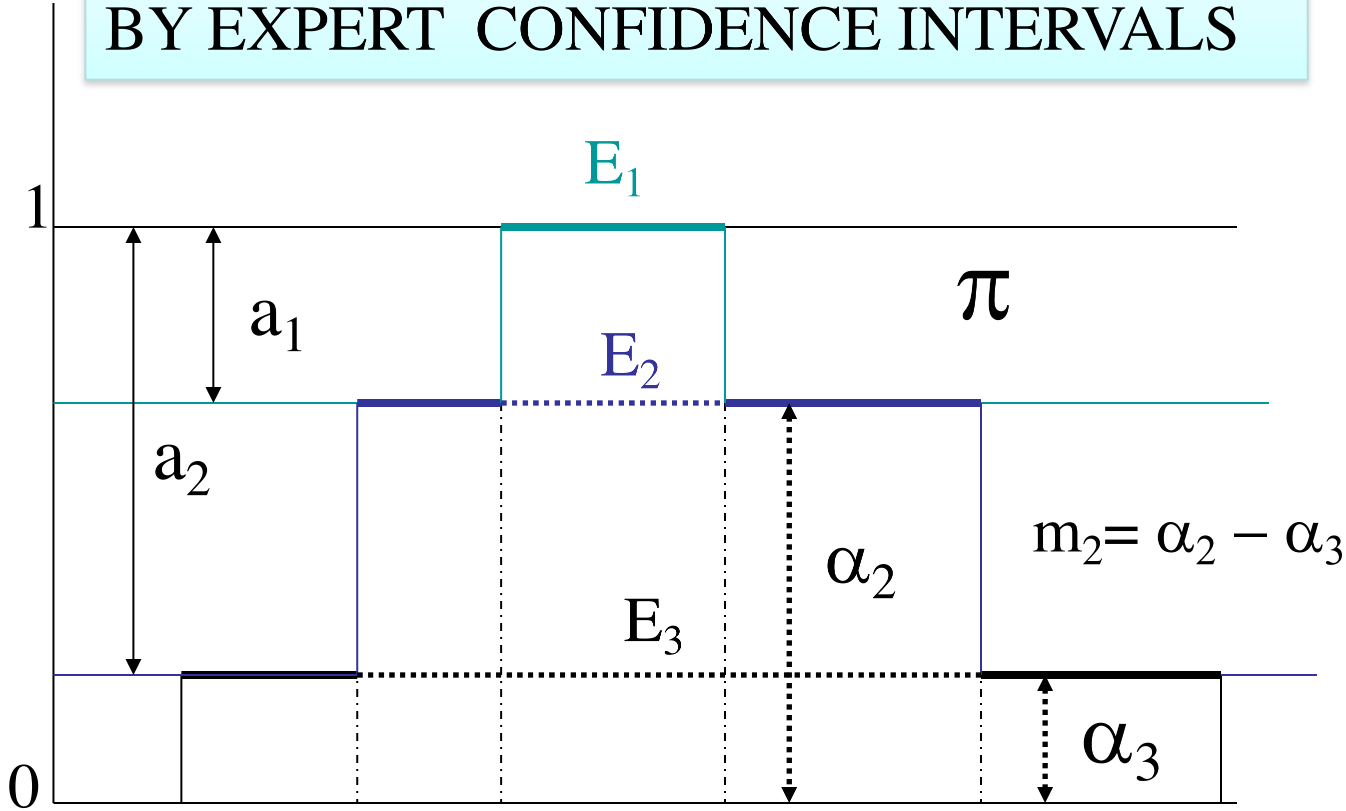
- Then $\mathcal{P}$ is representable by means of a possibility measure with distribution

$$\pi(x) = \min_{i = 1, \ldots n} \max (\mu_{Ei}(x), 1 - a_i)$$

- A consonant belief function with mass

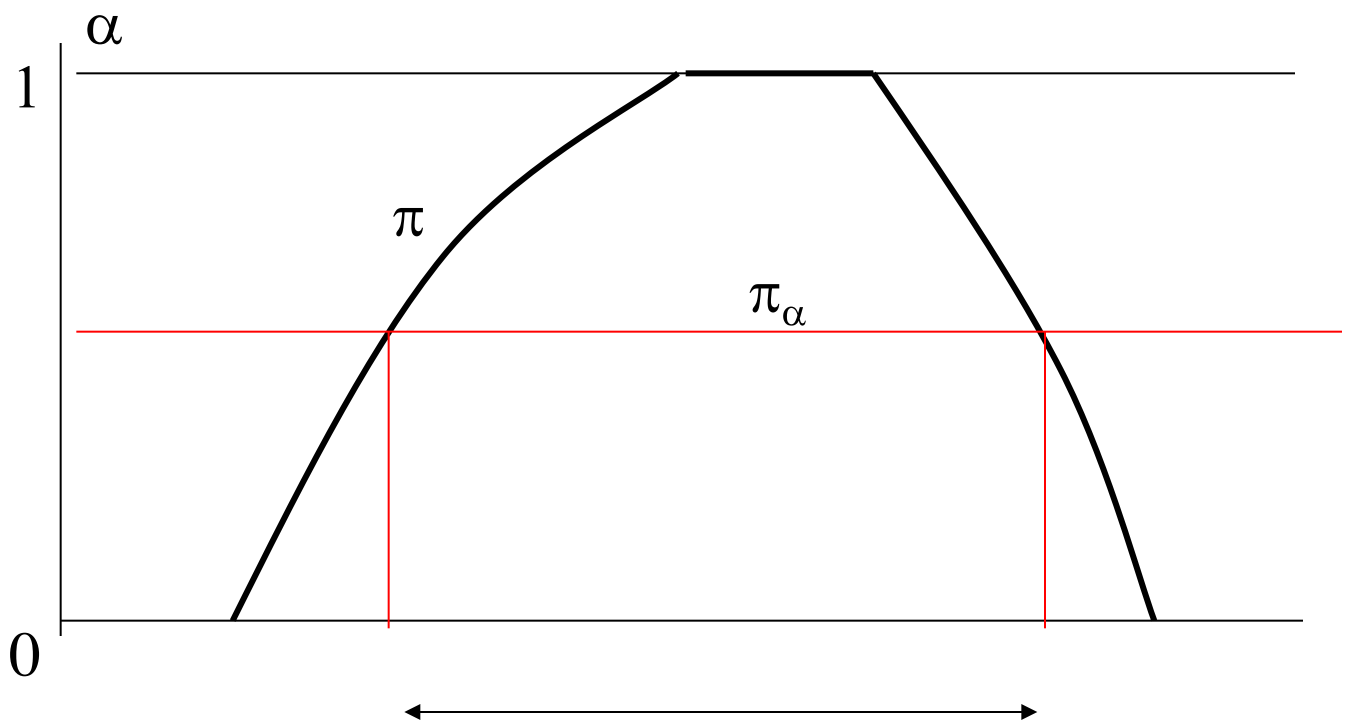
$$m(E_i) = a_i - a_{i+1}$$

POSSIBILITY DISTRIBUTION INDUCED BY EXPERT CONFIDENCE INTERVALS



*A **fuzzy interval** corresponds to :*

- *A convex set of probabilities:* $\{P: P(A_\alpha) \geq 1 - \alpha, \alpha \in (0, 1]\}$
- *A continuous belief function induced by the Lebesgue measure on [0, 1] i.e.,* $([0, 1], \mathcal{B}, \lambda) \to \mathbb{R}: \alpha \rightarrow \pi_\alpha$



FUZZY INTERVAL: $\Gamma(\alpha) = \pi_\alpha$

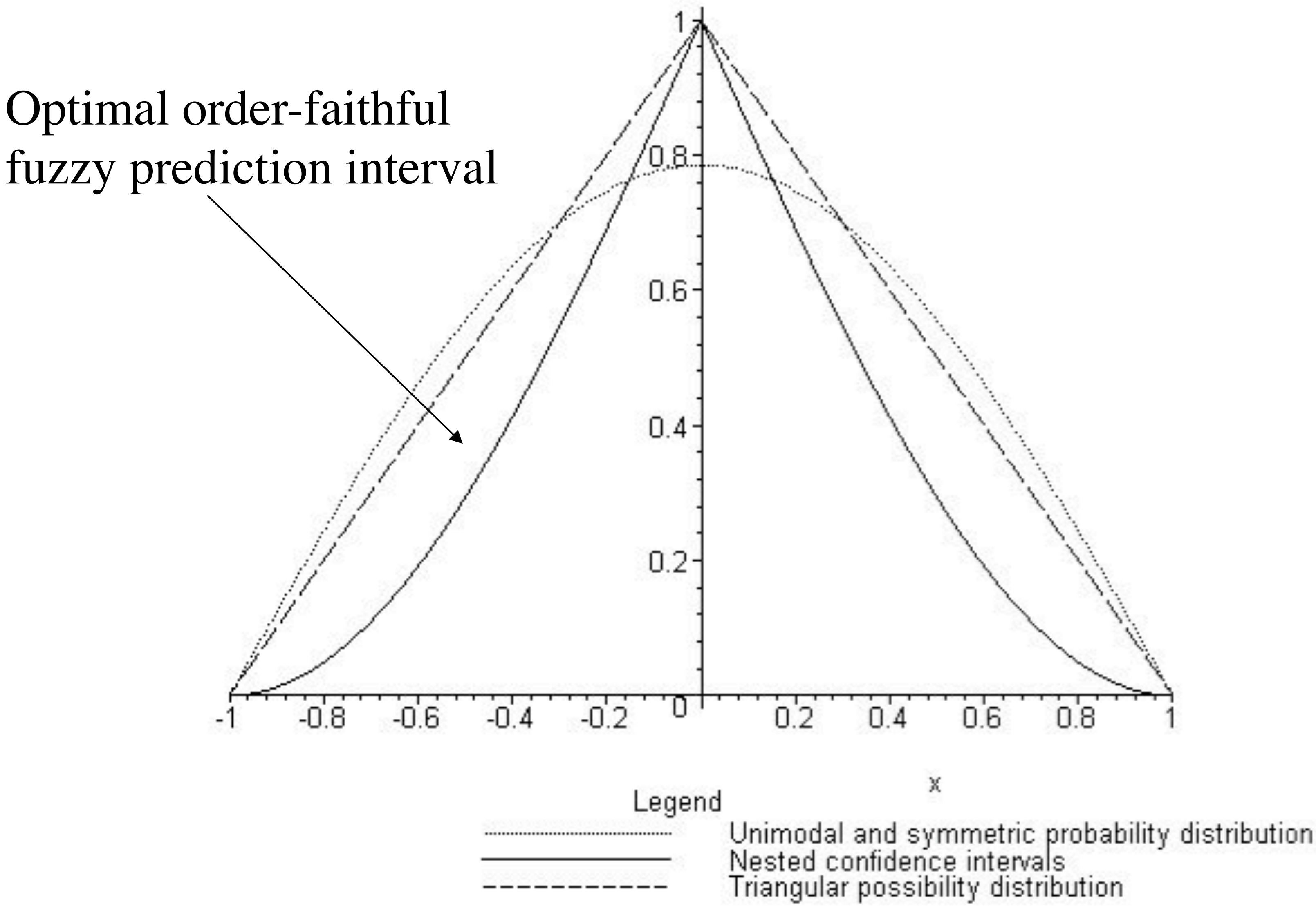
Possibilistic view of probabilistic inequalities

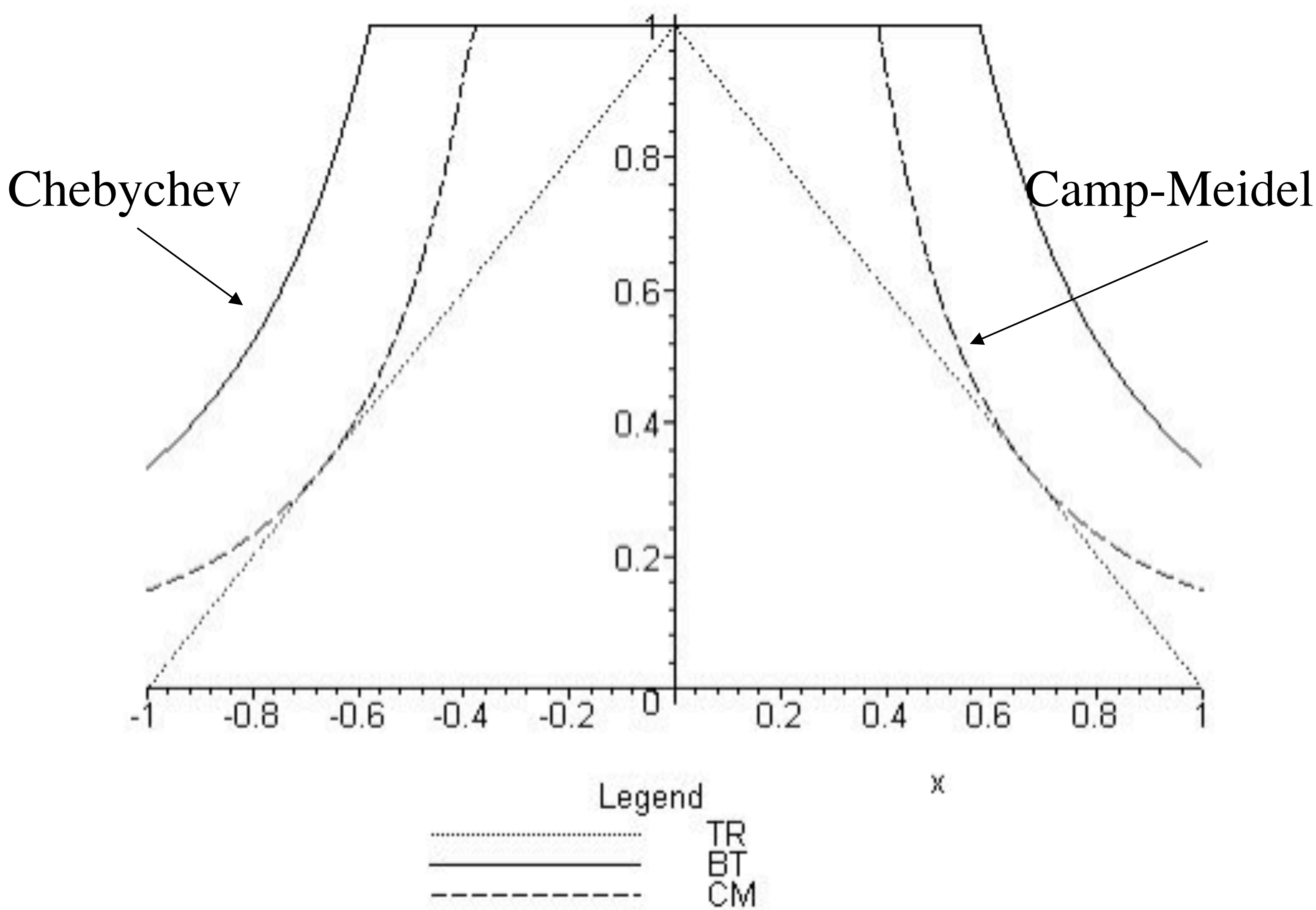
- Probabilistic inequalities can be used for representing consonant belief functions
  - Chebyshev inequality defines a possibility distribution that dominates *any* density with given mean and variance:

$$P(V \in [x^{mean} - k\sigma, x^{mean} + k\sigma]) \geq 1 - 1/k^2 \text{ is equivalent to}$$
$$\pi(x^{mean} - k\sigma) = \pi(x^{mean} + k\sigma) = 1/k^2$$

  - A triangular fuzzy number (TFN) defines a possibility distribution that dominates *any* unimodal density with the same mode and bounded support as the TFN (related to Gauss inequality, 1823).

$$P(V \in [x, 2x^{mode} - x]) \geq |x^{mode} - x|/a \text{ is equivalent to}$$
$$\pi(x) = 1 - |x^{mode} - x|/a \text{ on } [x^{mode} - a, x^{mode} + a]$$


Optimal order-faithful fuzzy prediction interval
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
x
Legend
Unimodal and symmetric probability distribution
Nested confidence intervals
Triangular possibility distribution



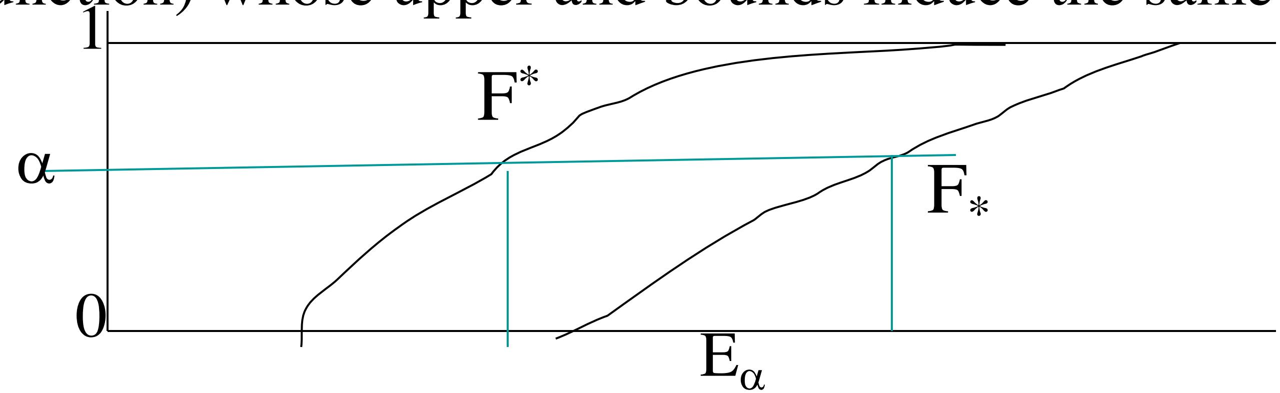
Chebychev
Camp-Meidel
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1
x
Legend
TR
BT
CM

Probability boxes

- Let be F_P be the cumulative distribution of P :

$$F_P(a) = P(-\infty, a]$$

- A set $\mathcal{P} = \{P: F^* \geq F_P \geq F_*\}$ induced by two cumulative disribution functions is called a **probability box (p-box)**,
- A p-box is a special random interval (continuous belief function) whose upper and bounds induce the same ordering.



Probability boxes from possibility distributions

- Given a fuzzy interval π
  - $F^*(a) = \Pi_M((-\infty, a]) = \pi(a)$ if $a \leq m$
    $= 1$ otherwise.
  - $F_*(a) = N_M((-\infty, a]) = 0$ if $a < m^*$
    $= 1 - \lim_{x \downarrow a} \pi(x)$ otherwise
- $\mathcal{P}(\pi)$ is a proper subset of $\mathcal{P} = \{P: F^* \geq F_P \geq F_*\}$
  - Not all P in $\mathcal{P}$ are such that $\Pi \geq P$

Representing families of probabilities by fuzzy intervals is more precise than with the corresponding pairs of PDFs

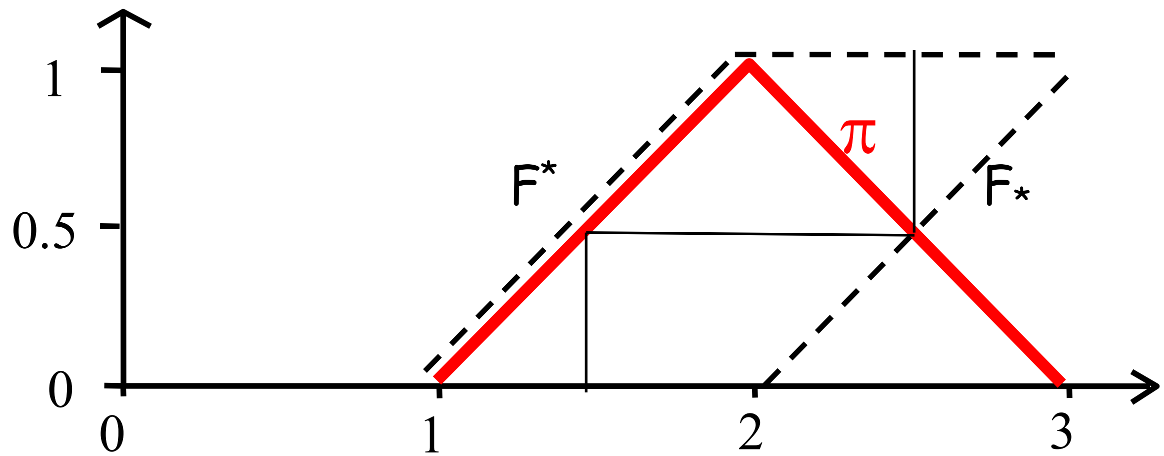
P-boxes vs. fuzzy intervals: counterexample

A triangular fuzzy number with support [1, 3] and mode 2.

Let P be defined by P({1.5})=P({2.5})=0.5.

Then $F_* < F < F$ but $P \notin P(\Pi)$ since

$P(\{1.5, 2.5\}) = 1 > \Pi(\{1.5, 2.5\}) = 0.5$



Practical representations of imprecise probabilities

- Real line
  - Fuzzy intervals (continuous consonant bf)
  - Probability boxes (continuous random intervals with comonotonic bounds)
- Discrete spaces
  - Probability intervals (2-monotone only)

How useful are these representations:

- P-boxes can address questions about threshold violations ($x \geq a$??),
- not questions of the form $a \leq x \leq b$
- The latter questions are better addressed by possibility distributions

LIKELIHOOD FUNCTIONS

- **Likelihood functions** $\lambda(x) = P(A| x)$ behave like possibility distributions when there is no prior on x, and $\lambda(x)$ is used as the likelihood of x.
- It holds that $\lambda(B) = P(A| B) \leq \max_{x \in B} P(A| x)$
- If $P(A| B) = \lambda(B)$ then λ should be set-monotonic:

$$\{x\} \subseteq B \text{ implies } \lambda(x) \leq \lambda(B)$$

It implies $\lambda(B) = \max_{x \in B} \lambda(x)$

Then it induces a (consonant) belief function

Maximum likelihood principle is possibility theory

- The classical coin example: θ is the unknown probability of "heads"
- Within n experiments: k heads, n-k tails
- P(k heads, n-k tails | θ) = $\theta^k \cdot (1-\theta)^{n-k}$ is

the degree of possibility $\pi(\theta)$ that the probability of "head" is θ.

In the absence of other information the best choice is the one that maximizes $\pi(\theta)$, $\theta \in [0, 1]$

It yields $\theta = k/n$.

LANDSCAPE OF UNCERTAINTY THEORIES

BAYESIAN/STATISTICAL PROBABILITY: the language of *unique* probability distributions (*Randomized points)*

UPPER-LOWER PROBABILITIES : the language of *disjunctive convex sets of probabilities, and lower expectations*

SHAFER-SMETS BELIEF FUNCTIONS: The language of Moebius masses (*Random disjunctive sets)*

QUANTITATIVE POSSIBILITY THEORY : The language of possibility distributions (*Fuzzy (nested disjunctive) sets)*

BOOLEAN POSSIBILITY THEORY (modal doxastic logic KD) : The language of *Disjunctive sets*

Important theoretical issues

- Relative **informativeness**.
- **Conditioning** : several definitions for several purposes.
- **Independence notions**: distinguish between epistemic and objective notions.
- Find a general setting for **information fusion** operations (e.g. Dempster rule of combination).
- **Uncertainty propagation :** Monte-carlo+ interval analysis (not possible for imprecise probabilities)

Comparing belief functions in terms of informativeness

- **Consonant case** : relative specificity.

π' more specific (more informative) than π in the wide sense if and only if $\pi' \leq \pi$.

(any possible value in information state π' is at least as possible in information state π)

  - Complete knowledge: $\pi(s_0) = 1$ and $= 0$ otherwise.
  - Ignorance: $\pi(s) = 1, \forall s \in S$

Comparing belief functions in terms of informativeness

- *<u>Using contour functions</u>*:

$$\pi(s) = Pl(s) = \sum_{x \in E} m(E)$$

m_1 is more cf-informative than m_2 iff $\pi_1 \leq \pi_2$

- *<u>Using belief or plausibility functions</u>* :

m_1 is more pl-informative that m_2 iff $Pl_1 \leq Pl_2$

iff $Bel_1 \geq Bel_2$

It corresponds to comparing credal sets $P(m)$:

$Pl_1 \leq Pl_2$ if and only if $P(m_1) \subseteq P(m_2)$

Comparing commonalities

- m_1 is **more Q-informative** than m_2 iff $Q_1 \leq Q_2$

where $Q(A) = \sum_{A \subseteq E} m(E)$ (antimonotonic)

- Denoted by $m_1 \sqsubseteq_Q m_2$
- *Q is all the larger as m has large focal sets (so m is less informative).*
- Typical information ordering for belief functions used by Smets

Specialisation

- m_1 is <u>*more specialised*</u> than m_2 if and only if there is a joint mass function $w(E, F)$ whose marginals are $m_1(E)$ and $m_2(F)$, such that each $w(E, F) = 0$ if E is not included in F.
- we have that
  - $m_1(E) = \sum_{E \subseteq F} w(E, F)\, m_2(F)$
  - Any focal set of m_1 is inside at least one focal set of m_2
  - Any focal set of m_2 contains at least one focal set of m_1
  - masses of focal sets of m_2 are shared among focal sets of m_1 that lie in them.

Results

- $m_1 \subseteq_s m_2$ implies $m_1 \subseteq_{Pl} m_2$ implies $m_1 \subseteq_{cf} m_2$
- $m_1 \subseteq_s m_2$ implies $m_1 \subseteq_Q m_2$ implies $m_1 \subseteq_{cf} m_2$
- *However $m_1 \subseteq_{Pl} m_2$ and $m_1 \subseteq_Q m_2$ are not comparable and can contradict each other*
- $m_1 \subseteq_{Pl} m_2$ and $m_2 \subseteq_Q m_1$ imply $\pi_1 = \pi_2$ ($m_1 =_{cf} m_2$)
- In the consonant case : all orderings collapse to $m_1 \subseteq_{cf} m_2$

Example

- $S = \{a, b, c\}$;
- $m_1(ab) = 0.5$, $m_1(bc) = 0.5$;
- $m_2(abc) = 0.5$, $m_2(b) = 0.5$
- Neither $m_1 \subseteq_s m_2$ nor $m_2 \subseteq_s m_1$ hold
- $m_2 \subset_{Pl} m_1 : Pl_1(A) = Pl_2(A)$
  but $Pl_2(ac) = 0.5 < Pl_1(ac) = 1$
- $m_1 \subset_Q m_2 : Q_1(A) = Q_2(A)$
  but $Q_1(ac) = 0 < Q_2(ac) = 0.5$
- And contour functions are equal : a/0.5, b/1, c/0.5

Conclusion

- *There exist a coherent range of set-functions in uncertainty theories combining interval and probability representations*.
  - Imprecise probability is the most general setting
  - The choice between subtheories depends on how expressive it is necessary to be in a given application.
  - There exist simple practical representations of belief functions, like possibility distributions and p-boxes (not interval probability masses!)
  - possibility theory is the simplest framework, and belief functions are a good compromise between calculability and expressivity

Uncertainty theories : distinct languages

- *Discrepancies between the theories remain on conditioning, combination rules, because their language primitives differ*:
  - Possibility theory : fuzzy sets of possible values
  - Belief functions : Moebius masses
  - Imprecise probabilities : Convex probability sets
- *It is not always clear how to express notions of one theory in the language of another :* One cannot obviously express concepts defined by Moebius masses using credal sets, and conversely, let alone possibility distributions.

Practical representation issues

- Lower probabilities are difficult to represent ($2^{|S|}$ values): The corresponding family is a polyhedron with potentially $|S|!$ vertices.
- Finite random sets (belief functions) are simpler but potentially $2^{|S|}$ values
- Possibility measures are simple ($|S|$ values) but sometimes not expressive enough.
- ***There is a need for simple and more expressive representations of imprecise probabilities or random sets.***

Conclusion: the nature of imprecise models

- ***Imprecise modeling is unusual*.**
  - In classical approaches, a probabilistic model is an approximate but precise representation of variability.
- In contrast, an imprecise model is of higher order, *hence is not objective*.

It models reality and knowledge (of an agent) about reality : reality must just be consistent with the imprecise representation, which depends on the agent.

- There is a need to reconsider the foundations of systems analysis in this perspective.

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A pioneer of possibility theory

- In the 1950's, **G.L.S. Shackle** called "degree of potential surprize" of an event its degree of impossibility = $1 - \Pi(A)$.

G. L. S. Shackle. Decision, Order, and Time in Human Affairs. Cambridge University Press, UK, 1961.

- Potential surprize is valued on a disbelief scale, namely a positive interval of the form [0, y*], where y* denotes the absolute rejection of the event to which it is assigned, and 0 means that nothing opposes to the occurrence of A.

- The degree of surprize of an event is the degree of surprize of its least surprizing realization.
- He introduces a notion of conditional possibility