Distances and conflict between belief functions

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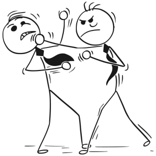
Purpose of the lecture

1. Provide an overview about distance and conflict measures between belief functions
2. Review basic geometry and consistency concepts
3. Discuss some properties of measures and illustrate comparative behaviours on maritime use cases

Distance? Conflict?



We will discuss measures which quantify how much two belief functions are:

distant, dissimilar, inconsistent, conflicting, orthogonal, in disagreement, . . .

and distinguish between distance and conflict measures.

Applications making use of distance or conflict measures

1. Quality assessment
   1.1 Accuracy
   1.2 Credibility
   1.3 Information loss

   These quality measures are either used as final measures of performance or reintroduced in the algorithm for further processing

2. Decision criterion
   2.1 Classification/identification solution
   2.2 Belief function approximation
   2.3 Evidential pattern matching
   2.4 Evidential information retrieval
   2.5 . . .

Outline

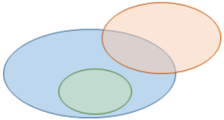
Outline

Reminder

Belief functions extend both:

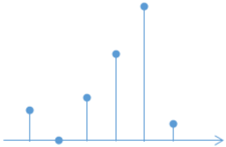
classical sets

- A categorical belief function is such that $m(A) = 1$ for some $A \subseteq X$ and defines the classical subset A



probabilities

- A Bayesian belief function is such that $m(A) > 0$ only for $|A| = 1$ (singleton elements) and defines a probability distribution over X



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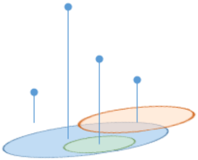

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Notations

- X is the frame of discernment of cardinality n: $X = \{x_1, \ldots, x_n\}$
- $\mathcal{P}(X)$ is its power set of cardinality 2^n: $\mathcal{P}(X) = \{\emptyset, x_1, \ldots, x_n, (x_1, x_2), \ldots, X\}$
- x is an element of X: $x \in X$
- A is a subset of X, element of $\mathcal{P}(X)$: $A \subseteq X$, $A \in \mathcal{P}(X)$
- $|A|$ is the cardinality of A
- $\overline{A}$ is the complement of A relatively to X: $\overline{A} = X \backslash A$
- $A \cap B$ denotes the intersection of A and B
- $A \cup B$ denotes the union of A and B
- $\mathcal{F} = \{A \subseteq X; m(A) \neq 0\}$ is the set of focal sets of m
- $|\mathcal{F}|$ is the number of focal sets of m

Outline

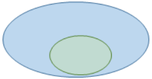
Preamble
- Interaction between sets
- Belief space and linear transformation

Basic interaction between sets

Inclusion

$$Inc(A, B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases}$$

- Inc is not symmetric



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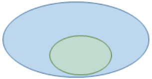
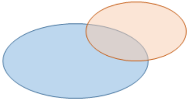


Intersection

$$Int(A,B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- The dual index of *Int* is $1 - Int$
- *Int* is symmetric



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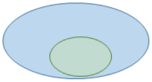
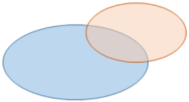


Intersection

$$Int(A, B) = \begin{cases} 1 & \text{if } A \cap B \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

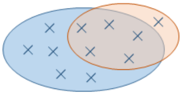
- The dual index of *Int* is $1 - Int$
- *Int* is symmetric



Jaccard

$$Jac(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

- *Jac* is a similarity measure between sets
- *Jac* is symmetric, positive
- $Jac(A, B) = 1$ iff $A = B$



Other interaction degrees

- $S(A, B) =$ any similarity index between sets
  For instance, Sørensen-Dice:
  $$\frac{2|A \cap B|}{|A| + |B|}$$
- $S_p(A, B)$ the specialisation matrix
- Pignistic $BetP(A, B) = \frac{|A \cap B|}{|B|}$
- Fixsen-Mahler:
  $$\frac{\phi(A \cap B)}{\phi(A)\phi(B)}$$
  where $\phi(A) = 1$ if $A \neq \varnothing$ and $\phi(\varnothing) = 0$ is a consistency measure, ϕ can be replaced by p, a probability measure over X
- Inclusion degree [Martin, 2012]
- ...

Index writing (1)

Belief	$Bel(A) = \sum_{B \subseteq A} m(B)$	$= \sum_{B \subseteq X} m(B) Inc(B, A)$
Plausibility	$Pl(A) = \sum_{A \cap B \neq \varnothing} m(B)$	$= \sum_{B \subseteq X} m(B) Int(A, B)$
Commonality	$q(A) = \sum_{A \subseteq B} m(B)$	$= \sum_{B \subseteq X} m(B) Inc(A, B)$
Contour function	$Pl(\{x\}) = \sum_{x \cap B \neq \varnothing} m(B)$	$= \sum_{B \subseteq X} m(B) Int(x, B)$
Pignistic probability	$BetP(A) = \sum_{B \subseteq X} m(B) \frac{\lvert A \cap B \rvert}{\lvert B \rvert}$	$= \sum_{B \subseteq X} m(B) Bet(A, B)$

- Uniform writing with set interaction inside the sum

Index writing (2)

For two BPAs, m_1 and m_2, Dempster's conflict can be written:

$$m_{12}(\varnothing) = \sum_{A \cap B = \varnothing} m_1(A)m_2(B) = \sum_{A,B \subseteq X} m_1(A)m_2(B)(1 - Int(A,B))$$

Dempster's agreement is:

$$\sum_{A,B \subseteq X} m_1(A)m_2(B)Int(A,B)$$

Outline

- Preamble
  - Interaction between sets
  - Belief space and linear transformation

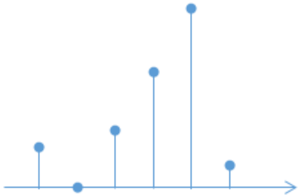
Remarkable vector spaces (1)

$\mathbb{R}^n$ vector space

The vector space $\mathbb{R}^n$ is the n-dimensional space where $\mathbb{F} = \mathbb{R}$ and

- vectors are represented by a list of n real numbers
- $\{e_1, \dots, e_n\}$ forms as basis for $\mathbb{R}^n$, where e_i is the i^{th} column of the identity matrix

- vector addition is $\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{pmatrix}$
- scalar multiplication is defined by $\alpha.\mathbf{v} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}$



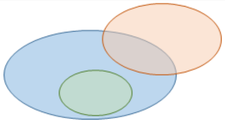
Remarkable vector spaces (2)

$\mathcal{P}(X)$ vector space

The power set of X, $\mathcal{P}(X)$, forms a vector space over the two-element field $\{0,1\}$ with $A, B \in \mathcal{P}(X)$:

- vectors are represented by a list of n binary numbers: $A = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$
- $\{x_1, \ldots, x_n\}$ is as basis for $\mathcal{P}(X)$, where x_i is the i^{th} element of X

- vector addition is the symmetric difference $A + B = (A \cup B) \backslash (A \cap B)$,
- scalar multiplication is defined by $1.A = A$ and $0.A = \varnothing$



Belief space (1)

Belief space

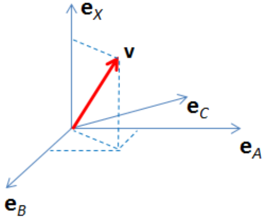
The *belief space*, denoted by $\mathcal{E}_X$, is the 2^n-dimensional vector space spanned by the set of vectors $\{\mathbf{e}_A, A \subseteq X\}$ over the field $\mathbb{R}$

- $\{\mathbf{e}_A, A \subseteq X\}$ defines a basis for $\mathcal{E}_X$
- $\mathbf{e}_A$ corresponds to a categorical mass function focused on A, $m(A) = 1$ and defines the classical set A

- Any vector $\mathbf{v}$ of $\mathcal{E}_X$ can be then written as:

$$\mathbf{v} = \sum_{A \subseteq X} \alpha_A \mathbf{e}_A$$

- $\alpha_A \in \mathbb{R}$ is the *coordinate* of $\mathbf{v}$ along the dimension $\mathbf{e}_A$



Belief space (2)

Basic Probability Assignment (BPA) vector

The *BPA vector* or *mass vector* $\mathbf{m}$ is the 2^n-dimensional vector whose coordinates $m(A)$ are such that:

$$\sum_{A \subseteq X} m(A) = 1$$

$$0 \leq m(A) \leq 1$$

$m(\emptyset) = 0$ under the closed-world assumption

We can write

$$\mathbf{m} = \sum_{A \subseteq X} m(A) \mathbf{e}_A$$

Remarkable vectors in $\mathcal{E}_X$

Belief vector

$$\mathbf{Bel} = \sum_{A \subseteq X} Bel(A)\mathbf{e}_A$$

Contour vector

$$\mathbf{pl} = \sum_{x \in X} pl(x)\mathbf{e}_x$$

Plausibility vector

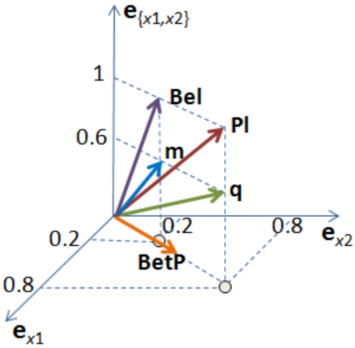
$$\mathbf{Pl} = \sum_{A \subseteq X} Pl(A)\mathbf{e}_A$$

Pignistic vector

$$\mathbf{BetP} = \sum_{A \subseteq X} Betp(A)\mathbf{e}_A$$

Commonality vector

$$\mathbf{q} = \sum_{A \subseteq X} q(A)\mathbf{e}_A$$



Index matrices (1)

To each index degree defined previously, we associate the matrix whose elements are the corresponding indexes/degrees between two subsets A and B of X, *i.e.*, two dimensions of $\mathcal{E}_X$. For example, for $N = 2$ and omitting the $\varnothing$ dimension, we have:

$$\mathbf{Inc} = \begin{array}{c} \\ \{x_1\} \\ \{x_2\} \\ \{x_1, x_2\} \end{array} \begin{array}{c} \begin{array}{ccc} \{x_1\} & \{x_2\} & \{x_1, x_2\} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{Int} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{Bet} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{Jac} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \quad \ldots$$

- Note that $\mathbf{Int}' = \mathbf{Int}$, $\mathbf{Jac}' = \mathbf{Jac}$, ... for the symmetric matrices

Index matrices (2)

Also, we can define rectangular matrices such as:

$$\mathbf{Bet}_x = \begin{array}{c} \\ \{x_1\} \\ \{x_2\} \end{array} \begin{array}{c} \begin{array}{ccc} \{x_1\} & \{x_2\} & \{x_1, x_2\} \end{array} \\ \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \end{array}$$

$$\mathbf{Int}_x = \begin{array}{c} \\ \{x_1\} \\ \{x_2\} \end{array} \begin{array}{c} \begin{array}{ccc} \{x_1\} & \{x_2\} & \{x_1, x_2\} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

- They act as projections over $\mathcal{E}_x$, the subspace of $\mathcal{E}_X$ built from the singleton elements

Linear transformations in the belief space

If we take m as the basic function, then the standard other functions can be retrieved by linear transformations of **m**:

$$
\begin{aligned}
\mathbf{Bel} &= \mathbf{Inc}'.\mathbf{m} \\
\mathbf{Pl} &= \mathbf{Int}.\mathbf{m} \\
\mathbf{Pl}_x &= \mathbf{Int}_x.\mathbf{m} \\
\mathbf{q} &= \mathbf{Inc}.\mathbf{m} \\
\mathbf{BetP} &= \mathbf{Bet}.\mathbf{m} \\
\mathbf{BetP}_x &= \mathbf{Bet}_x.\mathbf{m}
\end{aligned}
$$

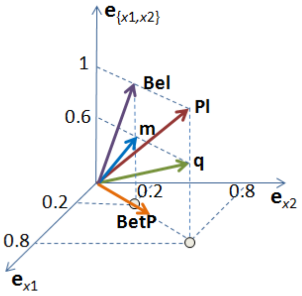
Example

$$X = \{x_1, x_2\}$$

$$\mathbf{m} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix}$$

$$\mathbf{Bel} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 1 \end{pmatrix}$$

$$\mathbf{Pl} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \\ 1 \end{pmatrix}$$



Outline

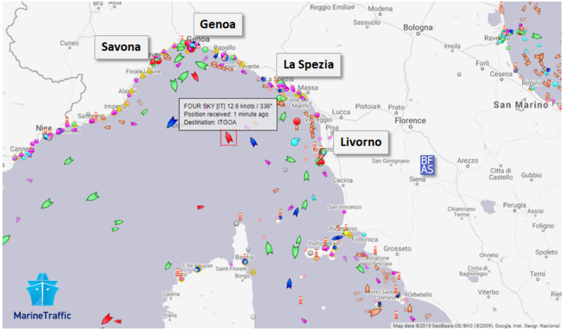
Example: Vessel destination prediction

$$X = \{x_1, x_2, x_3, x_4\} = \{\textsc{Savona}, \textsc{Genoa}, \textsc{La Spezia}, \textsc{Livorno}\}$$

Sources:

- S_1 AIS **destination** field
- S_2 **Position** relatively to the maritime route
- S_3 Ports historical visits by **type** & **flag**
- S_4 Ports capacity based on **size**
- S_5 Operator based on vessel history

AIS = Automatic Identification System



Observations about distances between belief functions

Example

$$X = \{x_1, x_2, x_3, x_4\} = \{\text{Savona}, \text{Genoa}, \text{La Spezia}, \text{Livorno}\}$$

$$m_5(\{x_1, x_2\}) = 0.8 \qquad m_1(\{x_1, x_2, x_3\}) = 0.8 \qquad m_2(\{x_4\}) = 0.8$$
$$m_5(X) = 0.2 \qquad m_1(X) = 0.2 \qquad m_2(X) = 0.2$$

- Because $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$ and $\{x_1, x_2\} \cap \{x_4\} = \varnothing$, we expect m_5 to be closer to m_1 than to m_2:
$$d(m_5, m_1) < d(m_5, m_2)$$
- However, neither m_1 nor m_2 share any focal element with m_5 (except X)
- The nature of belief functions requires that the interaction between focal elements is considered in the distance measure

Outline

- Distances between belief functions
  - Distance induced by a norm
  - Inner product

Distance induced by a norm

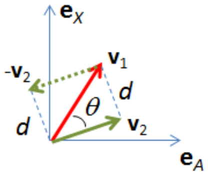
Given a normed space $(\mathcal{V}, \|.\|_W)$ the (metric) distance between $\mathbf{v}_1$ and $\mathbf{v}_2$ can be defined as the norm of their difference

Distance in $\mathcal{E}_X$

A norm $\|.\|_W$ defined over $\mathcal{E}_X$ induces a *distance* on $\mathcal{E}_X$ by:

$$d_W(\mathbf{m}_1, \mathbf{m}_2) = \|\mathbf{m}_1 - \mathbf{m}_2\|_W$$

- W denotes some interaction between focal sets



Examples of **W**

$$\mathbf{W} = \mathbf{U}'.\mathbf{U}$$

W	Def.
I	$I(A,B) = 1$ iff $A = B$
$\mathbf{Inc}\mathbf{Inc}'$	$Inc(A,B) = 1$ iff $A \subseteq B$
$\mathbf{Int}'\mathbf{Int}$	$Int(A,B) = 1$ iff $A \cap B \neq \varnothing$
$\mathbf{Int}'_x\mathbf{Int}_x$	$Int_x(x,B) = 1$ iff $x \in B$
$\mathbf{Bet}'\mathbf{Bet}$	$Bet(A,B) = \frac{\lvert A \cap B \rvert}{\lvert B \rvert}$
Jac	$J(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}$
S	$S(A,B)$ any similarity measure
$F(\mathbf{S}, R)$	F reward-penalty function

Metric distance definition

Metric

A function $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a *(full) metric* if and only if d satisfies the following properties for all $(y, z, t) \in \mathcal{V}^3$:

(d1) Positivity: $d(y, z) \geq 0$,

(d2) Symmetry: $d(y, z) = d(z, y)$,

(d3) Definiteness: $d(y, z) = 0 \Leftrightarrow y = z$,

- (d3)' Reflexivity (or identity): $d(y, y) = 0$
- (d3)" Separability: $d(y, z) = 0 \Rightarrow y = z$,

(d4) Triangle inequality: $d(y, z) \leq d(y, t) + d(z, t)$

(d1) together with (d3) define positive definiteness

Metric properties

			Metric	Semi-metric	Quasi-metric	Pseudo-metric	Semi-pseudo-metric	Pre-metric
(d1)	**Positivity**	$d(y,z) \geq 0$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
(d2)	**Symmetry**	$d(y,z) = d(z,y)$	$\times$	$\times$		$\times$	$\times$	
(d3)	**Definiteness**	$d(y,z) = 0 \Leftrightarrow y = z$	$\times$	$\times$	$\times$			
(d3)'	Reflexivity	$d(y,y) = 0$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
(d3)"	Separability	$d(y,z) = 0 \Rightarrow y = z$	$\times$	$\times$	$\times$			
(d4)	**Triangle inequ.**	$d(y,z) \leq d(y,t) + d(t,z)$	$\times$		$\times$	$\times$		

Minkowski family in $\mathcal{E}_X$

The Minkowski L_p norm is defined as

$$\|\mathbf{m}\|_W^{(p)} = \left(\left[(\mathbf{Um})^{\frac{p}{2}} \right]' \left[(\mathbf{Um})^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}$$

Minkowski family in $\mathcal{E}_X$

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L_p family in $\mathcal{E}_X$

The Minkowski (or L_p) family of distances between two belief functions induced by Minkowski norm can be written under the following general form:

$$d_W^{(p)}(m_1, m_2) = \left(\left[(\mathbf{Um}_1 - \mathbf{Um}_2)^{\frac{p}{2}}\right]' \left[(\mathbf{Um}_1 - \mathbf{Um}_2)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$$

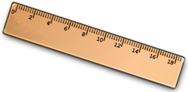
- $\mathbf{U}$ is the upper triangular matrix such that $\mathbf{W} = \mathbf{U}'\mathbf{U}$
- p is an integer higher than 1
- A normalisation constant should be added that will be omitted in the following

L_p interpretation

L_1 distance
Taxicab norm or Manhattan distance

The distance a car would drive in a city laid out in square blocks

L_2 distance
Euclidean distance



The distance obtained if measured with a ruler: the "intuitive" idea of distance

L_∞ distance
Chebyshev distance

The minimum number of moves kings require to travel between two squares on a chessboard

Ref. Wikipedia

L_2 - Euclidean distances

L_2 family

When $p = 2$, $d_W^{(p)}(m_1, m_2)$ becomes:

$$d_W^{(2)}(m_1, m_2) = \sqrt{[\mathbf{U}(\mathbf{m}_1 - \mathbf{m}_2)]'[\mathbf{U}(\mathbf{m}_1 - \mathbf{m}_2)]} = \sqrt{(\mathbf{m}_1 - \mathbf{m}_2)'\mathbf{W}(\mathbf{m}_1 - \mathbf{m}_2)}$$

where $\mathbf{W} = \mathbf{U}'\mathbf{U}$ is a positive semidefinite matrix.

- $d_W^{(2)}(m_1, m_2)$ is induced by the inner product $\otimes_W(m_1, m_2) = \mathbf{m}_1'\mathbf{W}\mathbf{m}_2$
- The most "intuitive" notion of distance: Length of a straight line between two points
- Generalisation of Pythagore theorem

Classical L_2 distances in $\mathcal{E}_X$ (1)

- Euclidean distance between BPAs: $\mathbf{W} = \mathbf{I}$

$$d_I^{(2)}(m_1, m_2) = \sqrt{(\mathbf{m}_1 - \mathbf{m}_2)'(\mathbf{m}_1 - \mathbf{m}_2)}$$

- Euclidean distance between belief functions: $\mathbf{W} = \mathbf{Inc}\mathbf{Inc}' = \mathbf{Inc}'\mathbf{2}$

$$d_{Inc'2}^{(2)}(m_1, m_2) = \sqrt{(\mathbf{Bel}_1 - \mathbf{Bel}_2)'(\mathbf{Bel}_1 - \mathbf{Bel}_2)}$$

- Euclidean distance between plausibility functions: $\mathbf{W} = \mathbf{Int}'\mathbf{Int} = \mathbf{Int2}$

$$d_{Int2}^{(2)}(m_1, m_2) = \sqrt{(\mathbf{Pl}_1 - \mathbf{Pl}_2)'(\mathbf{Pl}_1 - \mathbf{Pl}_2)}$$

Because $Pl(A) = 1 - Bel(\overline{A})$,

$$d_{Int2}^{(2)}(m_1, m_2) = d_{Inc'2}^{(2)}(m_1, m_2)$$

- $d_I^{(2)}$, $d_{Int2}^{(2)}$ and obviously $d_{Inc'2}^{(2)}$ are full metric distances

Classical L_2 distances in $\mathcal{E}_X$ (2)

- Euclidean distance between pignistic probabilities: $\mathbf{W} = \mathbf{Bet}'\mathbf{Bet} = \mathbf{Bet2}$

$$d^{(2)}_{Bet2}(m_1, m_2) = \sqrt{(\mathbf{BetP}_1 - \mathbf{BetP}_2)'(\mathbf{BetP}_1 - \mathbf{BetP}_2)}$$

- Euclidean distance between pignistic probabilities of singletons: $\mathbf{W} = \mathbf{Bet}'_x\mathbf{Bet}_x = \mathbf{Betx2}$

$$d^{(2)}_{Betx2}(m_1, m_2) = \sqrt{(\mathbf{BetP}_{x1} - \mathbf{BetP}_{x2})'(\mathbf{BeP}_{x1} - \mathbf{BetP}_{x2})}$$

- $d^{(2)}_{Bet2}$ and $d^{(2)}_{Betx2}$ are pseudo-metric distances (separability non-respected)
- Other functions of **Um** can be considered as well (see [Elouedi etal, 2001])

Jaccard L_2 Distance

Jaccard L_2 distance

When $\mathbf{W} = \mathbf{Jac}$:

$$d_J^{(2)}(m_1, m_2) = \sqrt{(\mathbf{m}_1 - \mathbf{m}_2)'\mathbf{Jac}(\mathbf{m}_1 - \mathbf{m}_2)}$$

where

$$\mathbf{Jac} = \frac{|A \cap B|}{|A \cup B|}$$

- **Jac** quantifies the similarity between pairs of focal elements of m_1 and m_2
- **Jac** is positive definite
- $d_J^{(2)}$ is a full metric, guarantying that $d_J^{(2)}(m_1, m_2) = 0 \Rightarrow m_1 = m_2$

Example: Fishing Vessel identification

Is the vessel of Fishing Vessel? Yes/No $X = \{F, \neg F\}$

Sources after processing give the following BPAs:

$$\text{Radar (speed)} \rightarrow \quad \mathbf{m}_1 = \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix} \quad \text{SAR (length)} \rightarrow \quad \mathbf{m}_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix}$$

$$\text{AIS (type)} \rightarrow \quad \mathbf{m}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{AIS (length)} \rightarrow \quad \mathbf{m}_4 = \begin{pmatrix} 0 \\ 0.7 \\ 0.3 \end{pmatrix}$$

Example: L_2 distances

Example

Speed (Radar) and Length (SAR)

$$\mathbf{m}_1 = \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix} \quad \mathbf{m}_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix}$$

$$d_I^{(2)}(m_1, m_2) = 0.25$$
$$d_J^{(2)}(m_1, m_2) = 0.14$$

- We observe that $d_J < d_I$
- This is because d_J removes some part of the errors along $\mathbf{e}_A$ and $\mathbf{e}_B$ dimensions proportionally to the similarity between A and B

Extensions of $d_J^{(2)}$

1. Any similarity between sets and reward function [Diaz etal, 2006]

$$d_{F(J)}^{(2)}(m_1, m_2) = \sqrt{(\mathbf{m}_1 - \mathbf{m}_2)' F(\mathbf{S}, R)(\mathbf{m}_1 - \mathbf{m}_2)}$$

2. Ordered sets using Hausdorff distance [Sunberg & Rogers, 2013]

$$d_{Haus}^{(2)}(m_1, m_2) = \sqrt{(\mathbf{m}_1 - \mathbf{m}_2)' \mathbf{D}_H (\mathbf{m}_1 - \mathbf{m}_2)}$$

3. Continuous belief functions using intervals similarity [Attiaoui etal, 2013]

$$d_{\delta}^{(2)}(m_1, m_2) = \sqrt{\|f_1\|^2 + \|f_2\|^2 - 2 \otimes (f_1, f_2)}$$

Manhattan distances (1)

L_1 family

When $p = 1$, $d_W^{(p)}(m_1, m_2)$ becomes:

$$d_W^{(1)}(m_1, m_2) = \left[(\mathbf{Um}_1 - \mathbf{Um}_2)^{\frac{1}{2}}\right]' \left[(\mathbf{Um}_1 - \mathbf{Um}_2)^{\frac{1}{2}}\right]$$

- Manhattan distance is induced by the L_1 norm:

$$\|\mathbf{m}\|_W^{(1)} = \left[(\mathbf{Um})^{\frac{1}{2}}\right]' \left[(\mathbf{Um})^{\frac{1}{2}}\right]$$

- The L_1 norm is not induced by an inner product

Classical Manhattan distances in $\mathcal{E}_X$

- Manhattan distance between BPAs:

$$d_I^{(1)}(m_1, m_2) = \sum_{A \subseteq X} |m_1(A) - m_2(A)|$$

- Manhattan distance between belief functions

$$d_{Inc'}^{(1)}(m_1, m_2) = \sum_{A \subseteq X} |Bel_1(A) - Bel_2(A)| = d_{Int}^{(1)}(m_1, m_2)$$

- Manhattan distance between contour functions

$$d_{Intx}^{(1)}(m_1, m_2) = \sum_{x \in X} |Pl_1(x) - Pl_2(x)|$$

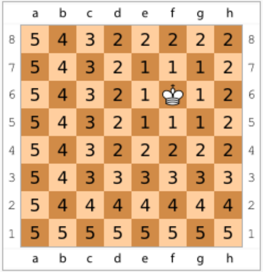
Chebyshev distances

L_∞ family

When $p = \infty$, $d_W^{(p)}(m_1, m_2)$ becomes:

$$d_W^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \{ |(\mathbf{U}\ \mathbf{m}_1)' \mathbf{e}_A - (\mathbf{U}\ \mathbf{m}_2)'\ \mathbf{e}_A| \}$$

- Different Chebyshev distances are obtained by changing the weighting matrix $\mathbf{U}$



Classical Chebyshev distances in $\mathcal{E}_X$

- Chebyshev distance between pignistic probabilities:

$$d_{Bet}^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \{ |(\mathbf{Bet\ m}_1)' \, \mathbf{e}_A - (\mathbf{Bet\ m}_2)' \, \mathbf{e}_A| \}$$

- Chebyshev distance between plausibility functions:

$$d_{Int}^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \{ |\mathbf{Pl}_1' \, \mathbf{e}_A - \mathbf{Pl}_2' \, \mathbf{e}_A| \}$$

- Chebyshev distance between the contour functions:

$$d_{Intx}^{(\infty)}(m_1, m_2) = \max_{x \in X} \{ |\mathbf{Pl}_1' \, \mathbf{e}_x - \mathbf{Pl}_2' \, \mathbf{e}_x| \}$$

Other distances

Helinger distance family

$$d_W^{(H)}(m_1, m_2) = \left(1 - \otimes_W^{\frac{1}{2}}(m_1, m_2)\right)^{\frac{1}{2}}$$

Information-based distances family

$$d_U(m_1, m_2) = |U(m_1) - U(m_2)|$$

where U is any uncertainty measure defined for belief functions. For instance [Denoeux, 2001],

$$d_{GC}(m_1, m_2) = |(\mathbf{m}_1 - \mathbf{m}_2)'\mathbf{c}_A|$$

where $\mathbf{c}_A$ is the column vector of cardinality

Belief-Interval distance [Han, Dezert, Yang, 2014]

$$d_{BI}^{(E)}(m_1, m_2) = \sqrt{\sum_{A \subseteq X} d_{Wa}^2([Bel_1(A), Pl_1(A)], [Bel_2(A), Pl_2(A)])}$$

where d_{Wa} is Wassertein distance

Outline

- Distances between belief functions
  - Distance induced by a norm
  - Inner product

Inner product in $\mathcal{E}_X$

Inner product belief space

Let us consider the inner product between two BPAs m_1 and m_2 in $\mathcal{E}_X$ of the following general form:

$$\begin{aligned}\otimes_W(\mathbf{m}_1, \mathbf{m}_2) &= \mathbf{m}_1' \mathbf{W} \mathbf{m}_2 \\ &= (\mathbf{U}\mathbf{m}_1)'(\mathbf{U}\mathbf{m}_2)\end{aligned}$$

$\mathcal{E}_X$ endowed with $\otimes_W$ is an inner product space

- $\mathbf{W} = \mathbf{U}'\mathbf{U}$ is a weighting matrix, symmetric and positive definite
- $\mathbf{U}$ is a upper triangle matrix, for instance
- When $\mathbf{W} = \mathbf{I}$, $\otimes_I$ is the standard dot product

Orthogonality

Orthogonality

Two vectors of $\mathcal{E}_X$ (*i.e.*, two belief functions) are *orthogonal*, and we note $\mathbf{m}_1 \perp_W \mathbf{m}_2$, iff their inner product is null:

$$\mathbf{m}_1 \perp_W \mathbf{m}_2 \Leftrightarrow \otimes_W(\mathbf{m}_1, \mathbf{m}_2) = 0$$

- This is a way to quantify how much $\mathbf{m}_1$ differs from $\mathbf{m}_2$
- $\otimes_W(\mathbf{m}_1, \mathbf{m}_2)$ quantifies a notion of "agreement" : The higher $\otimes_W(\mathbf{m}_1, \mathbf{m}_2)$, the more in agreement $\mathbf{m}_1$ and $\mathbf{m}_2$
- $f\left(\otimes_W(\mathbf{m}_1, \mathbf{m}_2)\right)$ quantifies a notion of "disagreement" for any decreasing function f

Examples of inner products in the belief space

$$\otimes_W(\mathbf{m}_1, \mathbf{m}_2) = \mathbf{m}_1' \mathbf{W} \mathbf{m}_2$$

Notation	W	Def.
$\otimes_I^s$	$\mathbf{I}$	$I(A,B) = 1$ iff $A = B$
$\otimes_{Inc}^s$	$\mathbf{Inc}\mathbf{Inc}'$	$Inc(A,B) = 1$ iff $A \subseteq B$
$\otimes_{Int2}^s$	$\mathbf{Int}'\mathbf{Int}$	$Int(A,B) = 1$ iff $A \cap B \neq \emptyset$
$\otimes_{Intx}^s$	$\mathbf{Int}_x'\mathbf{Int}_x$	$Int_x(x,B) = 1$ iff $x \in B$
$\otimes_{Bet}^s$	$\mathbf{Bet}'\mathbf{Bet}$	$Bet(A,B) = \frac{\|A \cap B\|}{\|B\|}$
$\otimes_J^s$	$\mathbf{Jac}$	$J(A,B) = \frac{\|A \cap B\|}{\|A \cup B\|}$
$\otimes_S^s$	$\mathbf{S}$	$S(A,B)$ any similarity measure
$\otimes_{F(S)}^s$	$F(\mathbf{S}, R)$	F reward-penalty function

Norm induced by an inner product

Norm induced by an inner product

An inner product $\otimes_W$ over $\mathcal{E}_X$ induces a norm $\|\mathbf{m}\|_W$ over $\mathcal{E}_X$ defined as:

$$\|\mathbf{v}\|_W = \sqrt{\otimes_W(\mathbf{m}, \mathbf{m})}$$

- Not every norm arises from an inner product
- Only the norms satisfying the parallelogram law
- For instance, L_2-norm is induced by an inner product while L_1-norm is not

Cosine family

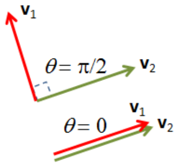
The cosine measure defines a normalised inner product measure of similarity between belief functions

Cosine between belief functions

The general formulation for a cosine measure in $\mathcal{E}_X$ is:

$$\cos_W(m_1, m_2) = \frac{\mathbf{m}_1' \mathbf{W}\ \mathbf{m}_2}{\|\mathbf{m}_1\|_W \cdot \|\mathbf{m}_2\|_W}$$

- Because $m(A) \geq 0$ for all $A \subseteq X$, for all $(m_1, m_2) \in \mathcal{E}_X$, we have $0 \leq \theta \leq \frac{\pi}{2} \Rightarrow 0 \leq \cos_W(\theta) \leq 1$ with:
  - $\cos_W(m_1, m_2) = 0$ iff $\mathbf{m}_1 \perp_W \mathbf{m}_2$ (orthogonal)
  - $\cos_W(m_1, m_2) = 1$ iff $\mathbf{m}_1 = \alpha \mathbf{m}_2$ (collinear)
- A dissimilarity measure in $\mathcal{E}_X$ can be obtained by: $\cos^d_W(m_1, m_2) = 1 - \cos_W(m_1, m_2)$



Cosine plausibility

Example

Length (SAR) and type (AIS):

$$\mathbf{m}_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix} \quad \mathbf{m}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We have: $\otimes_I(Pl_2, Pl_3) = 1.8$. The cosine measure is:

$$\begin{aligned} \cos_{Int2}(m_2, m_3) &= \frac{\mathbf{m}_2' \mathbf{Int}^2\ \mathbf{m}_3}{\|\mathbf{Intm}_2\|.\|\mathbf{Intm}_3\|} = \frac{\mathbf{Pl}_2'\mathbf{Pl}_3}{\|\mathbf{Pl}_2\|.\|\mathbf{Pl}_3\|} \\ &= \frac{1.8}{\sqrt{2.64 \times 2}} = 0.7833 \end{aligned}$$

- We obtain a normalised value

Dempster's conflict (1)

Dempster's conflict can be put under the form of an inner product as:

$$\otimes^{d}_{Int}(m_1, m_2) = \mathbf{m}_1'(1 - \mathbf{Int})\,\mathbf{m}_2$$

- **Int** is the matrix of intersection indexes
- $\otimes^{d}_{Int}(m_1, m_2)$ should thus not be called an inner product because $(1 - \mathbf{Int})$ is neither positive nor definite

We can also write:

$$\begin{aligned}\otimes^{d}_{Int}(m_1, m_2) &= 1 - \mathbf{m}_1'\,\mathbf{Int}\,\mathbf{m}_2 \\ &= 1 - \otimes^{s}_{Int}(m_1, m_2)\end{aligned}$$

where $\otimes^{s}_{Int}(m_1, m_2)$ is Dempster's agreement

Dempster's agreement

Example

Length (SAR) and type (AIS): $\mathbf{m}_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix}$ $\mathbf{m}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{aligned} \otimes^{s}_{Int}(\mathbf{m}_2, \mathbf{m}_3) &= \mathbf{m}_2' \ \mathbf{Int} \ \ \mathbf{m}_3 \\ &= \begin{pmatrix} 0.2 & 0 & 0.8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.8 \end{aligned}$$

- The agreement is not maximum (the conflict is not null) because $m_2(\{F\})$ and $m_3(\{\neg F\})$ are not null AND $Int(F, \neg F) = 0$

Dempster's conflict (2)

- **Int** defines a similarity measure over $\mathcal{P}(X)$
- $1 - \mathbf{Int}$ defines a dissimilarity measure over $\mathcal{P}(X)$
- Unfortunately, this does not imply that $\otimes^{d}_{Int}$ is a dissimilarity measure in $\mathcal{E}_X$
- In particular, $\otimes^{d}_{Int}(\mathbf{m}, \mathbf{m}) = 0$ is not satisfied (reflexivity property)
- Any increasing function of $\otimes^{d}_{Int}$ can be used to define a "distance". For instance [Ristic, Smets 2006]:

$$d_{RS}(m_1, m_2) = -\log(1 - \otimes^{d}_{Int}(m_1, m_2))$$

Outline

Outline

Consistency of sets

Definition

- A is *consistent* iff $A \neq \varnothing$
- A is *inconsistent* iff $A = \varnothing$

Properties

- $0 \leq \phi(A) \leq 1$
- Minimum iff A is totally inconsistent, maximum iff A is totally consistent

Measure

- $\begin{cases} \phi(A) = 0 \text{ iff } A = \varnothing \\ \phi(A) = 1 \text{ iff } A \neq \varnothing \end{cases}$

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Measure

- $\begin{cases} \phi(A) = 0 \text{ iff } A = \varnothing \\ \phi(A) = 1 \text{ iff } A \neq \varnothing \end{cases}$

- Two sets A and B are consistent iff $A \cap B \neq \varnothing$ and *inconsistent* iff $A \cap B = \varnothing$
- N sets $\{A_n\}_{n=1}^{N}$ are consistent iff $\bigcap_{n=1,\dots N} A_n \neq \varnothing$

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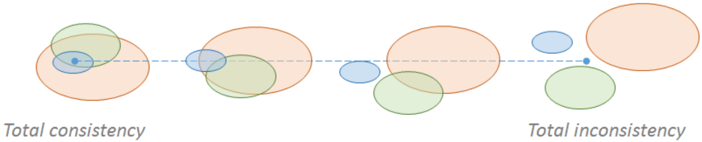

Consistency of belief functions (Definitions)

Total inconsistency

A mass function m is *totally inconsistent* iff $m(\emptyset) = 1$.

While the state of *total inconsistency* is uniquely characterised, different definitions characterise the state of **total consistency**:

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A mass function m is *probabilistically consistent* iff

$$\forall A \in \mathcal{F}, A \neq \varnothing$$

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A mass function m is *probabilistically consistent* iff

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Logical

A mass function m is *logically consistent* iff

$$\bigcap_{A \in \mathcal{F}} A \neq \varnothing$$

Consistency of belief functions (definitions)

Total inconsistency

A mass function m is *totally inconsistent* iff $m(\varnothing) = 1$.

While the state of *total inconsistency* is uniquely characterised, different definitions characterise the state of **total consistency**:

Probabilistic

A mass function m is *probabilistically consistent* iff

$$\forall A \in \mathcal{F}, A \neq \varnothing$$

Pairwise

A mass function m is *pairwise consistent* iff

$$\forall (A, B) \in \mathcal{F}^2, A \cap B \neq \varnothing$$

Logical

A mass function m is *logically consistent* iff

$$\bigcap_{A \in \mathcal{F}} A \neq \varnothing$$

N-consistency of belief functions

The following family of consistency definitions has recently been proposed:

N-consistency

A mass function m is said to be *N-consistent*, with $1 \leq N \leq |\mathcal{F}|$, iff $\forall \{A_n\}_{n=1}^{N} \subseteq \mathcal{F}$, we have

$$\bigcap_{n=1,\ldots,N} A_n \neq \varnothing$$

The family encompasses the classical definitions as particular cases:

- Probabilistic consistency coincides with the 1-consistency
- Pairwise consistency coincides with the 2-consistency
- Logical consistency coincides with the $|\mathcal{F}|$-consistency

Consistency measures properties

Consistency measure

A consistency measure ϕ should satisfy the following properties:

(cs1) Bounded: $\phi_{\text{min}} \leq \phi(m) \leq \phi_{\text{max}}$

(cs2) Extreme consistent values:

$$\phi(m) = \phi_{\text{min}} \iff m \text{ totally inconsistent} \iff m(\emptyset) = 1$$
$$\phi(m) = \phi_{\text{max}} \iff m \text{ totally consistent}$$

- (cs2) depends on the definition of *total consistency*
- Common to set $\phi_{\text{min}} = 0$ and $\phi_{\text{max}} = 1$

Consistency measures

Probabilistic consistency

[Destercke & Burger, 2013]

$$\phi_1(m) = 1 - m(\varnothing)$$

Pairwise consistency

[Yager, 1992]

$$\phi_2(m) = \sum_{A \cap B \neq \varnothing} m(A)m(B)$$

Logical consistency

[Destercke & Burger, 2013]

$$\phi_\pi(m) = \max_{x \in X} pl(x).$$

Consistency measures

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- ϕ_1 satisfies (cs1) and (cs2) for the definition of 1-consistency (probabilistic consistency)
- ϕ_2 satisfies (cs1) and (cs2) for the definition of 2-consistency (pairwise consistency)
- ϕ_π satisfies (cs1) and (cs2) for the definition of $|\mathcal{F}|$-consistency (logical consistency)

Refining consistency of destination predictions

Example

S_2 (Track-to-route algo.):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_5 (VTS operator):

$$\begin{cases} m_5(x_1, x_2) = 0.8 \\ m_5(x_3) = 0.1 \\ m_5(x_4) = 0.1 \end{cases}$$

- m_2 and m_5 are equally consistent according to ϕ_1 and ϕ_π:

$$\phi_1(m_2) = \phi_1(m_5) = 1 \qquad \phi_\pi(m_2) = \phi_\pi(m_5) = 0.8$$

- They can be discriminated considering the pairwise intersection of the focal sets:

$$\phi_2(m_2) = 0.92 \text{ and } \phi_2(m_5) = 0.66$$

Consistency measures

Probabilistic consistency

[Destercke & Burger, 2013]

$$\phi_1(m) = 1 - m(\varnothing)$$

Pairwise consistency

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[Yager, 1992]

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Logical consistency
[Destercke & Burger, 2013]

$$\phi_\pi(m) = \max_{x \in X} pl(x).$$

The N-consistency of a mass function m defined over X is, for $1 \leq N \leq |\mathcal{F}|$, defined by

$$\phi_N(m) = 1 - m^{(N)}(\varnothing)$$

where $m^{(N)} = m^{(N-1)} \textcircled{\cap} m$, with $m^{(0)} = m_X$.

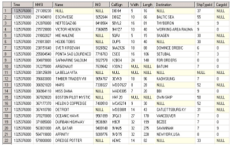
- The family ϕ_N is ordered $\phi_1(m) \geq \phi_2(m) \geq \ldots \geq \phi_{|\mathcal{F}|}$
- Measures ϕ_N satisfy properties (cs1) and (cs2) according to the definition of N-consistency
- $\phi_{|\mathcal{F}|}$ is an alternative measure of logical consistency to ϕ_π

Consistent destination predictions

Example

S_1 (AIS destination): $\begin{cases} m_1(x_1) = 0.8 \\ m_1(\emptyset) = 0.2 \end{cases}$

- SAVONA has the closest name matching the World Port Index
- SAVOONGA (Alaska region) is a possible match



AIS destination field manually fed, needs correction

Consistent destination predictions

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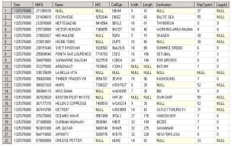


AIS destination field manually fed, needs correction

		m_1		$m_1^{(2)}$		$m_1^{(3)}$	
		$(x_1, 0.8)$	$(\emptyset, 0.2)$	$(x_1, 0.64)$	$(\emptyset, 0.36)$	$(x_1, 0.51)$	$(\emptyset, 0.49)$
m_1	$(x_1, 0.8)$	x_1	$\emptyset$	x_1	$\emptyset$	x_1	$\emptyset$
	$(\emptyset, 0.2)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
		$\phi_1(m_1) = 0.8$		$\phi_2(m_1) = 0.64$		$\phi_3(m_1) = 0.51$	

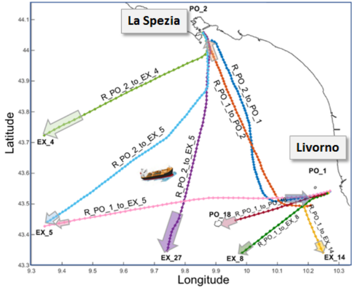
Consistent destination predictions

Example

$$S_2 \text{ (Track-to-route algo.)}: \begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2, x_4) = 0.3 \\ m_2(x_3, x_4) = 0.1 \end{cases}$$

$$\begin{cases} \phi_1(m_2) = 1 - m_2(\emptyset) = \phi_2(m_2) = 1 - m_2^{(2)}(\emptyset) = 1 \\ \phi_3(m_2) = 1 - m_2^{(3)}(\emptyset) = 0.89 \end{cases}$$

- m_2 is 1-consistent (probabilistically consistent) and 2-consistent (pairwise consistent)
- m_2 is not 3-consistent (*i.e.*, not logical consistent since $|\mathcal{F}_2| = 3$)



Maritime routes extracted from AIS data from Jan 1 - Feb 20 2013) 2013 [Pallotta etal, 2013]

Monotonic consistency of destination predictions

Example

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S_5 (VTS operator):

$$\begin{cases} m_5(x_1, x_2) = 0.8 \\ m_5(x_3) = 0.1 \\ m_5(x_4) = 0.1 \end{cases}$$

	$\phi_1(m)$	$\phi_2(m)$	$\phi_{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$
m_2	1	0.92	0.88	0.8
m_5	1	0.66	0.51	0.8

- ϕ_π does not seem to belong to the family ϕ_N

Several shades of consistency

Monotonic N-consistency measure

The monotonic N-consistency of a mass function m defined over X is, for $N \geq 0$, defined by

$$\psi_N(m) = \left(1 - m^{(N)}(\varnothing)\right)^{\frac{1}{N}}$$

where $m^{(N)} = m^{(N-1)} \textcircled{\cap} m$, with $m^{(0)} = m_X$.

- $\phi_1(m) = \psi_1(m) \geq \psi_2(m) \geq \ldots \geq \psi_{|\mathcal{F}|}(m) \geq \phi_\pi(m) = \lim_{N \to \infty} \psi_N(m)$
- Measures ψ_N satisfy properties (cs1) and (cs2)
- The family ψ_N is bounded by the measures of probabilistic and logical consistency
- $\psi_{|\mathcal{F}|}$ is an alternative measure of logical consistency to ϕ_π

Monotonic consistency of destination predictions

Example

S_2 (Track-to-route algo.):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_5 (VTS operator):

$$\begin{cases} m_5(x_1, x_2) = 0.8 \\ m_5(x_3) = 0.1 \\ m_5(x_4) = 0.1 \end{cases}$$

	$\phi_1(m)$	$\phi_2(m)$	$\phi_{\lvert\mathcal{F}\rvert}(m)$	$\phi_\infty(m)$
m_2	1	0.92	0.88	0.8
m_5	1	0.66	0.51	0.8

	$\psi_1(m)$	$\psi_2(m)$	$\psi_{\lvert\mathcal{F}\rvert}(m)$	$\psi_\infty(m)$
m_2	1	0.96	0.958	0.8
m_5	1	0.812	0.801	0.8

Outline

- Consistency and conflict between belief functions
  - Consistency and inconsistency
  - Conflict between belief functions

Conflict definitions

Total conflict

Two mass functions m_1 and m_2 are said to be *totally conflicting* if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, where $\mathcal{C}_i = \cup_{A \in \mathcal{F}_i} A$ denote the disjunction of the focal sets of m_i.

Different definitions characterise the state of **non-conflict**:$\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

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Different definitions characterise the state of **non-conflict**:$\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

1-non-conflict

m_1 and m_2 are *1-non-conflicting* iff $\forall A \in \mathcal{F}_{12}$, we have $A \neq \emptyset$

2-non-conflict

m_1 and m_2 are *2-non-conflicting* iff $\forall (A, B) \in \mathcal{F}_{12}^2$, we have $A \cap B \neq \emptyset$

$\mathcal{F}_{12}$-non-conflict

m_1 and m_2 are *$\mathcal{F}_{12}$-non-conflicting* iff

$$\bigcap_{A \in \mathcal{F}_{12}} A \neq \emptyset$$

Desirable conflict properties

(cf1)	**Boundedness**	$\kappa_{\text{min}} \leq \kappa(m_1, m_2) \leq \kappa_{\text{max}}$
(cf2)'	**Extreme min. value**	κ_{min} iff m_1 and m_2 minimally consistent
(cf2)"	**Extreme max. value**	κ_{max} iff m_1 and m_2 maximally consistent
(cf3)	**Symmetry**	$\kappa(m_1, m_2) = \kappa(m_2, m_1)$
(cf4)	**Insensitivity to refinement**	$\kappa(m_1, m_2) = \kappa(m_{\rho(1)}, m_{\rho(2)})$
(cf5)	**Imprecision monotonicity**	$m_1 \sqsubseteq_s m_1' \Rightarrow \kappa(m_1, m_2) \geq \kappa(m_1', m_2)$
(cf6)	**“Ignorance is bliss”**	$\kappa(m_X, m) = 1 - \phi(m)$

Conflict measures

Inconsistency-based measure of conflict

The conflict between m_1 and m_2 can be defined as the inconsistency of their conjunctive combination:

$$\kappa(m_1, m_2) = 1 - \phi(m_1 \circledcap m_2),$$

where ϕ is a measure of consistency.

- To each consistency measure previously defined, corresponds a conflict measure:

$$\kappa_1(m_1, m_2) = 1 - \phi_1(m_1 \circledcap m_2) = (m_1 \circledcap m_2)(\emptyset)$$
$$\kappa_\pi(m_1, m_2) = 1 - \phi_\pi(m_1 \circledcap m_2) = 1 - \max_{x \in X} pl_{1 \circledcap 2}(x)$$

These measures were shown to satisfy the properties (cf1) to (cf6), considering the different definitions non-conflict.

Several shades of conflict

N-conflict measure

The N-conflict between two mass functions m_1 and m_2 for $N \geq 0$, is defined by:

$$\kappa_N(m_1, m_2) = 1 - \left(1 - (m_1 \circledcirc m_2)^{(N)}(\varnothing)\right)^{\frac{1}{N}}$$

where $m^{(N)}$ denotes the N successive conjunctive combinations of m with itself.

- Monotonically ordered family of measures

$$\kappa_1(m_1, m_2) \leq \kappa_2(m_1, m_2) \leq \ldots \leq \kappa_{|\mathcal{F}_{12}|}(m_1, m_2) \leq \kappa_\pi(m_1, m_2) = \lim_{N \to \infty} \kappa_N(m_1, m_2)$$

- Encompasses existing measures of probabilistic and logical conflict
- Satisfy the properties (cf1) to (cf6), considering the different definitions of non-conflict

Outline

Outline

Conflict and distances
- A norm-based view of conflict
- Zoom on measures properties

Consistency as the distance to inconsistency

The state of total inconsistency is such that:

$$m(\emptyset) = 1 \Longleftrightarrow Pl(A) = 0,\ \forall A \subseteq X$$

We can prove that:

Consistency as a norm

$$\phi_1(m) = \max_{A \subseteq X} Pl(A) = \|m\|_{Int}^{(\infty)}$$

$$\phi_\pi(m) = \max_{x \in X} pl(x) = \|m\|_{Intx}^{(\infty)}$$

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Distance to inconsistency

$$\phi_1(m) = d_{Int}^{(\infty)}(m, m_\varnothing)$$

$$\phi_\pi(m) = d_{Intx}^{(\infty)}(m, m_\varnothing)$$

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Distance to inconsistency

$$\phi_1(m) = d_{Int}^{(\infty)}(m, m_\varnothing)$$

$$\phi_\pi(m) = d_{Intx}^{(\infty)}(m, m_\varnothing)$$

- The consistency of a mass function can be seen as its distance to the *totally inconsistent* knowledge state.

Conflict and distance

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state.

Conflict and distance

$$\begin{cases} \kappa_1(m_1, m_2) = 1 - d_{Int}^{(\infty)}\left(m_1 \textcircled{\cap} m_2, m_\emptyset\right) \\ \kappa_\pi(m_1, m_2) = 1 - d_{Intx}^{(\infty)}\left(m_1 \textcircled{\cap} m_2, m_\emptyset\right) \end{cases}$$

Conflict and distance

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state.

Conflict and distance

$$\begin{cases} \kappa_1(m_1, m_2) = 1 - d_{Int}^{(\infty)}(m_1 \circledcap m_2, m_\varnothing) \\ \kappa_\pi(m_1, m_2) = 1 - d_{Intx}^{(\infty)}(m_1 \circledcap m_2, m_\varnothing) \end{cases}$$

$$d^{(2)}(Pl_1, Pl_2) = \sqrt{\sum_{A \subseteq X} (\kappa_1(m_1, m_A) - \kappa_1(m_2, m_A))^2}$$

- $d^{(2)}(Pl_1, Pl_2)$ quantifies how much m_1 and m_2 are in conflict with the same sets (according to κ_1)

Combining distances and conflict

Distance and conflict are different notions and their corresponding measures reflect the corresponding semantics through sets of properties.

To capture the "discrepancy" between belief functions:

- A two-dimensional measure [Liu, 2006]:

$$\delta_L^{2D} = \left(\otimes_{Int}^{d}(m_1, m_2); d_{Bet}^{(\infty)}(m_1, m_2)\right)$$

  can be generalised to: $d_{(W,V)}^{2D}(m_1, m_2) = \left(\otimes_{W}^{d}(m_1, m_2); d_V(m_1, m_2)\right)$

- Product [Martin, 2012]:

$$\delta(m_1, m_2) = (1 - \mathrm{Inc}(m_1, m_2)).d(m_1, m_2)$$

  with $\mathrm{Inc}(m_1, m_2) = \frac{1}{|\mathcal{F}_1|.|\mathcal{F}_2|} \sum_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} Inc(A, B)$ is an inclusion index

Outline

Conflict and distances

- A norm-based view of conflict
- Zoom on measures properties

Measure properties

			Distance measures	Conflict measures
(δ_1)	**Boundedness**	$\delta_{\text{min}} \leq \delta(m_1, m_2) \leq \delta_{\text{max}}$		$\times$
(δ_1)'	**Positivity**	$0 \leq \delta(m_1, m_2)$	$\times$	$\times$
(δ_2)'	**Extreme min. value**	δ_{min} iff m_1 and m_2 minimally distant / consistent	$\times$	$\times$
(δ_2)"	**Extreme max. value**	δ_{max} iff m_1 and m_2 maximally distant / consistent	$(\times)$	$\times$
(δ_3)	**Symmetry**	$\delta(m_1, m_2) = \delta(m_2, m_1)$	$\times$	$\times$
(δ_4)	**Insensitivity to refinement**	$\delta(m_1, m_2) = \delta(m_{\rho(1)}, m_{\rho(2)})$		$\times$
(δ_5)	**Imprecision monotonicity**	$m_1 \sqsubseteq_s m_1' \Rightarrow \delta(m_1, m_2) \geq \delta(m_1', m_2)$		$\times$
(δ_6)	**"Ignorance is bliss"**	$\delta(m_X, m) = 1 - \phi(m)$		$\times$
(δ_7)	**Reflexivity**	$\delta(m, m) = 0$	$\times$	
(δ_8)	**Separability**	$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$	$\times$	
(δ_9)	**Triangle inequality**	$\delta(m_1, m_2) \leq \delta(m_1, m_3) + \delta(m_3, m_2)$	$\times$	

Reflexivity in $\mathcal{E}_X$

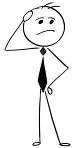
Reflexivity

$$\delta(m, m) = 0$$

- This property is generally not satisfied by conflict measures. For instance,

$$(m \circledcap m)(\varnothing) = \otimes^d_{Int}(m, m) \neq 0$$

Ex.: $m' = \begin{pmatrix} 0.2 & 0.2 & 0.6 \end{pmatrix}$, $\otimes^d_{Int}(m, m) = 0.08$



- Relaxing reflexivity allows to express a notion of "internal conflict"

We can distinguish between:

1. Two distinct but identical belief functions (e.g., from two - independent - sources)
2. The same belief function

Separability in $\mathcal{E}_X$

Separability

$$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$$

- Not required for conflict measures
- Satisfied by (full) metric measures
- Not satisfied by pseudo-metric measures

Conflict or distance? Which measure?

To select the proper measure, we have the following degrees of freedom:

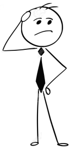
- The desirable properties (separability, reflexivity, "ignorance is bliss", etc) conveying notions of either distance or conflict
- The interaction between focal elements (weaker such as **Int**, **Inc**, or stronger such as **Jac**) and their meaning
- The meaning of the measure:
  - value of p in Minkowski family
  - angle versus distance, conflict versus distance, ...

Summary (1)

1. Basic Probability Assignments (BPAs) can be interpreted as vectors in the belief space
2. The belief, plausibility, commonality functions and the pignistic probability are linear transformations of the BPA vector
3. The belief space is an inner product space with the inner product $\mathbf{m}_1'\mathbf{W}\mathbf{m}_2$
4. The belief space is a normed space and the norm is defined by the L_p family of norms
5. Inner products (and cosines) measure the orthogonality between belief functions
6. Metrics measure the distance between belief functions
7. Conflict measures are not required to satisfy the basic metric properties of reflexivity and separability

Summary (2)

- ❽ Distances and conflict measures capture different notions of discrepancy between belief functions
- ❾ Conflict is defined as the inconsistency resulting from the conjunctive combination
- ❿ Several shades of conflict can be defined from gradually stronger notions of sets consistency
- ➔ Conflict depends on the dependency between sources: See Sébastien Destercke class from the 2015 BFTA school in Stella Plage (France)



References (1)

Distances

- A.-L. Jousselme and P. Maupin, "Distances in evidence theory: Comprehensive survey and generalizations," *International Journal of Approximate Reasoning*, vol. 53, p. 118–145, February 2012.
- A.-L. Jousselme, D. Grenier, and E. Bossé, "A new distance between two bodies of evidence," *Information Fusion*, vol. 2, p. 91–101, June 2001.
- M. Bouchard, A.-L. Jousselme, and P.-E. Doré, "A proof for the positive definiteness of the Jaccard index matrix," *International Journal of Approximate Reasoning*, vol. 54, no. 5, p. 615–626, 2013.

Conflict

- S. Destercke, T. Burger, "Toward an axiomatic definition of conflict between belief functions", *IEEE Trans. of Systems, Man and Cybernetics - Part B: Cybernetics*, 2013, vol 42, no. 2, p. 585-596.
- F. Pichon, A.-L. Jousselme, N. Ben Abdallah, "Several shades of conflict," *Fuzzy sets and systems*, Vol 366, July 2019, p 63-84.
- F. Pichon, A.-L. Jousselme, "A norm-based view of conflict" *Advances in Intelligent and Soft Computing*, Proc. of the 4th Int. Conf. on Belief Functions, 2016.
- N. Ben Abdallah, A.-L. Jousselme, and F. Pichon, "An ordered family of consistency measures of belief functions", In F. Cuzzolin, S. Destercke, T. Denœux and A. Martin, editors, Belief Functions: Theory and Applications, Proc. of the 5th Int. Conf., BELIEF 2018, Compiègne, France, Sep. 17-21, 2018, vol. 11069 of LNCS, p. 199-207, Springer, 2018.
- R. R. Yager. On considerations of credibility of evidence. International Journal of Approximate Reasoning, 7(1/2):45-72, 1992.

References (2)

Geometry of belief functions

- F. Cuzzolin, "A Geometric Approach to the Theory of Evidence," IEEE Transactions on Systems, Man, and Cybernetics - Part C: Applications and Reviews, vol. 38, no. 4, p. 522–534, 2008.
- F. Cuzzolin, "Geometry of Dempster's rule of combination," IEEE Transactions on Systems, Man, and Cybernetics - Part B: Cybernetics, vol. 34, no. 2, p. 961–977, 2008.

Matrix calculus of belief functions

- P. Smets, "The application of the matrix calculus to belief functions," *International Journal of Approximate Reasoning*, vol. 31, p. 1–30, 2002.

Maritime surveillance application

- A.-L. Jousselme, G. Pallotta, "Dissecting uncertainty handling techniques: Illustration on maritime anomaly detection", *Journal of Advances in Information Fusion*, 13(2), Dec. 2018.
- G. Pallotta, M. Vespe, K. Bryan, "Vessel Pattern Knowledge Discovery from AIS Data - A Framework for Anomaly Detection and Route Prediction", Entropy, 2013, 5, 2218-2245.
- N. Ben Abdallah, C. Iphar, G. Arcieri, A.-L. Jousselme, "Fixing errors in the AIS destination field", Proc. of the OCEANS 2019 conference, Marseille, France, June 2019.

References (3)

Other references used to prepare this presentation

- W. Liu, "Analyzing the degree of conflict among belief functions," *Artificial Intelligence*, vol. 170, p. 909–924, 2006.
- B. Ristic and P. Smets, "The TBM global distance measure for the association of uncertain combat ID declarations," *Information Fusion*, vol. 7, p. 276–284, 2006.
- Z. Sunberg and J. Rogers, "A belief function distance metric for orderable sets," *Information Fusion*, vol. 14, p. 361–373, 2013.
- J. Diaz, M. Rifqi, and B. Bouchon-Meunier, "A similarity measure between basic belief assignments," in *proc. of the 9th International Conference Information Fusion*, (Firenze, Italy), 2006.
- D. Attiaoui, P.-E. Doré, A. Martin, and B. Ben Yaghlane, "A distance between continuous belief functions," in *Proc. of the Scalable Uncertainty Management (SUM) conference*, vol. 7520 of Lecture Notes in Artificial Intelligence, p. 194–205, Springer-Verlag Berlin Heidelberg, 2012.
- Martin A. (2012) About Conflict in the Theory of Belief Functions. In: Denoeux T., Masson M.-H. (eds) Belief Functions: Theory and Applications. Advances in Intelligent and Soft Computing, vol 164. Springer, Berlin, Heidelberg.

Questions ?

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